ISEN 629 **Engineering Optimization**
Fall 2007

Homework # 4

Due Date: November 15, Thursday.

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1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a self-concordant function.

   (a) Write down the definition of a self-concordant function for the univariate case.

   (b) Assume that $f''(x) \neq 0$. Show that the self-concordance condition can be expressed as
   $$\left| \frac{d}{dx}\left(f''(x)^{-1/2}\right) \right| \leq 1.$$

   (c) Find the "extreme" self-concordant functions of one variable, *i.e.*, the functions f_1 and f_2 that satisfy
   $$\frac{d}{dx}\left(f_1''(x)^{-1/2}\right) = 1, \ \frac{d}{dx}\left(f_2''(x)^{-1/2}\right) = -1,$$
   respectively.

   (d) Show that either $f''(x) = 0$ for all $x \in \mathrm{dom}\ f$ or $f''(x) > 0$ for all $x \in \mathrm{dom}\ f$.

2. (a) Show that $f(x) = 1/x$ with domain $(0, 8/9)$ is self-concordant.

   (b) Show that the function
   $$f(x) = \alpha \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x}$$
   with $\mathrm{dom}\ f = \{x \in \mathbb{R}^n : a_i^T x < b_i, i = 1, \ldots, m\}$, is self-concordant if dom f is bounded and
   $$\alpha > (9/8) \max_{i=1,\ldots,m} \sup_{x \in \mathrm{dom}\ f} (b_i - a_i^T x).$$

3. (Composition with logarithm) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $\mathrm{dom}\, g = \{x \in \mathbb{R} : x > 0\}$ and
   $$|g'''(x)| \leq 3\frac{g''(x)}{x}$$
   for all x. Prove that $f(x) = -\ln(-g(x)) - \ln x$ is self-concordant on $\{x : x > 0, g(x) < 0\}$.

   (**Hint:** Use the inequality
   $$\frac{3}{2}rp^2 + q^3 + \frac{3}{2}p^2 q + r^3 \leq 1,$$
   which holds for $p, q, r \geq 0$ with $p^2 + q^2 + r^2 = 1$).

4. Prove that the following functions are self-concordant:

(a) $f(x, y) = -\ln(y^2 - x^T x)$ on $\{(x, y) : \|x\| < y\}$;

(b) $f(x, y) = -2\ln y - \ln(y^{2/p} - x^2)$, with $p \geq 1$, on $\{(x, y) \in \mathbb{R}^2 : |x|^p < y\}$;

(c) $f(x, y) = -\ln y - \ln(\ln y - x)$ on $\{(x, y) : e^x < y\}$.