ISEN 629: Engineering Optimization

Lecture 11

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Nonsmooth convex optimization

We consider a convex minimization problem

$$\begin{array}{ll} & \min f_0(x), \\ \text{s.t.} & f_i(x) \leq 0,\ i = 1, \ldots, m, \\ & x \in Q \subseteq \mathbb{R}^n, \end{array}$$

where Q is a closed convex set and $f_i(x), i = 0, \ldots, m$ are convex functions (do not need to be differentiable).

Example:

$$f(x) = \max_{1 \leq j \leq p} \phi_j(x),$$

where $\phi_j(x)$ are convex and differentiable.

General convex functions

Denote by

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n : |f(x)| < \infty\}$$

the domain of function f. We assume that $\mathrm{dom}\, f \neq \emptyset$.

Definition

Function $f(x)$ is called convex if $\mathrm{dom}\, f$ is convex and for any $x, y \in \mathrm{dom}\, f$ and $\alpha \in [0, 1]$ the following inequality holds:

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y).$$

We call f(x) concave if $-f(x)$ is convex.

General convex functions

Lemma (3.1.1, Jensen inequality)

For any $x_1, \ldots, x_m \in \mathrm{dom}\, f$ *and coefficients* $\alpha_1, \ldots, \alpha_m$ *such that*

$$\sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \geq 0, i = 1, \ldots, m, \tag{1}$$

we have

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \leq \sum_{i=1}^{m} \alpha_i f(x_i).$$

A point $x = \sum_{i=1}^{m} \alpha_i x_i$ with coefficients α_i satisfying (1) is called a convex combination of points $x_i, i = 1, \ldots, m$.

General convex functions

Corollary (3.1.1)

Let x be a convex combination of points $x_1, \dots, x_m$. Then

$$f(x) \le \max_{1\le i\le m} f(x_i).$$

Proof: Due to Jensen inequality, we have

$$f(x) = f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \le \sum_{i=1}^{m} \alpha_i f(x_i) \le \max_{1\le i\le m} f(x_i) \sum_{i=1}^{m} \alpha_i = \max_{1\le i\le m} f(x_i).$$

General convex functions

Corollary (3.1.2)

Denote by

$$\Delta = \mathrm{Conv}\{x_1, \dots, x_m\} = \left\{ x = \sum_{i=1}^{m} \alpha_i x_i : \sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \ge 0 \right\}.$$

Then

$$\max_{x\in\Delta} f(x) = \max_{1\le i\le m} f(x_i).$$

General convex functions

Theorem (3.1.1)

Function f is convex if and only if for any $x, y \in \mathrm{dom}\, f$ and $\beta \ge 0$ such that $y + \beta(y - x) \in \mathrm{dom}\, f$, we have

$$f(y + \beta(y - x)) \ge f(y) + \beta(f(y) - f(x)). \qquad (2)$$

Proof: $\boxed{\Rightarrow}$ Let f be convex. Denote by $\alpha = \frac{\beta}{1+\beta}$ and $u = y + \beta(y - x)$. Then

$$y = \frac{1}{1+\beta}(u + \beta x) = (1 - \alpha)u + \alpha x,$$

so $f(y) \le (1-\alpha)f(u) + \alpha f(x) = \frac{1}{1+\beta} f(u) + \frac{\beta}{1+\beta} f(x)$. Multiplying both sides by $1 + \beta$, we obtain (2).

General convex functions

$\boxed{\Leftarrow}$ Assume that for any $x, y \in \mathrm{dom}\, f$ and $\beta \ge 0$ such that $y + \beta(y - x) \in \mathrm{dom}\, f$, we have

$$f(y + \beta(y - x)) \ge f(y) + \beta(f(y) - f(x)).$$

Let us fix arbitrary $x, y \in \mathrm{dom}\, f$ and $\alpha \in (0, 1]$ and denote by

$$\beta = \frac{1-\alpha}{\alpha}; \quad u = \alpha x + (1-\alpha)y \Rightarrow x = \frac{1}{\alpha}(u - (1-\alpha)y) = u + \beta(u - y).$$

Applying the inequality for y, u, we have

$$f(x) \ge f(u) + \beta(f(u) - f(y)) = \frac{1}{\alpha} f(u) - \frac{1-\alpha}{\alpha} f(y).$$

Multiplying both sides by α, we obtain

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y).$$

General convex functions

Example: Consider the following function of two variables:

$$f(x,y) = \begin{cases} 0, & \text{if } x^2+y^2<1, \\ \phi(x,y), & \text{if } x^2+y^2=1, \end{cases}$$

where $\phi(x,y)$ is an arbitrary nonnegative function defined on the unit circle.

- $\mathrm{dom}\, f = \{(x,y) : x^2+y^2 \le 1\}$ - closed and convex.
- f is convex.
- No reasonable properties on the boundary of $\mathrm{dom}\, f$.
- To exclude such convex functions from consideration, we will introduce the notion of closed convex function.

General convex functions

Definition

A convex function f is called closed if its epigraph is a closed set.

Theorem (3.1.4)

If convex function f is closed then all its lower level sets are either empty or closed.

Proof: By definition,

$$\mathcal{L}_f(\beta) = \{x \in \mathbb{R}^n : f(x) \le \beta\} \Rightarrow (\mathcal{L}_f(\beta), \beta) = \mathrm{epi}(f) \cap \{(x,t) : t = \beta\}.$$

$\mathcal{L}_f(\beta)$ is closed and convex as an intersection of two closed convex sets. □

- If f is convex and continuous and its domain is closed, then f is a closed function.
- A closed convex function is not necessarily continuous.

General convex functions

Examples:

1. Linear function is closed and convex.
2. $f(x) = |x|, x \in \mathbb{R}^1$, is closed and convex since
$$\mathrm{epi}(f) = \{(x,t) : t \ge x, t \ge -x\}$$
is the intersection of two closed convex sets.
3. All differentiable and convex on $\mathbb{R}^n$ functions are closed convex functions.
4. Function $f(x) = 1/x, x > 0$, is convex and closed. Note that $\mathrm{dom}\, f$ is open.
5. Function $f(x) = \|x\|$, where $\|\cdot\|$ is any norm, is closed and convex, since for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$:
$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= \|\alpha x_1 + (1-\alpha)x_2\| \\ &\le \|\alpha x_1\| + \|(1-\alpha)x_2\| \\ &= \alpha\|x_1\| + (1-\alpha)\|x_2\|. \end{aligned}$$

Operations with convex functions

Theorem (3.1.5)

Let functions f_1 and f_2 be closed and convex, and let $\beta \ge 0$. Then all of the following functions are closed and convex:

1. $f(x) = \beta f_1(x)$, $\mathrm{dom}\, f = \mathrm{dom}\, f_1$.
2. $f(x) = f_1(x) + f_2(x)$, $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$.
3. $f(x) = \max\{f_1(x), f_2(x)\}$, $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$.

Proof:

1. $$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &\le \beta(\alpha f_1(x_1) + (1-\alpha) f_1(x_2)) \\ &= \alpha f(x_1) + (1-\alpha) f(x_2). \end{aligned}$$
2. It is easy to show that $f(x)$ is convex. To show that it is closed, consider a sequence $\{(x_k, t_k)\} \subset \mathrm{epi}(f)$:
$$t_k \ge f(x_k) = f_1(x_k) + f_2(x_k), \ \lim_{k\to\infty} x_k = \bar{x} \in \mathrm{dom}\, f, \ \lim_{k\to\infty} t_k = \bar{t}.$$

Operations with convex functions

Since f_1 and f_2 are closed, we have

$$\sup \lim_{k\to\infty} f_1(x_k) \geq f_1(\bar{x}), \sup \lim_{k\to\infty} f_2(x_k) \geq f_2(\bar{x}),$$

and

$$\bar{t} = \lim_{k\to\infty} t_k \geq \sup \lim_{k\to\infty} f_1(x_k) + \sup \lim_{k\to\infty} f_2(x_k) \geq f(\bar{x}).$$

Hence, $(\bar{x}, \bar{t}) \in \mathrm{epi}(f)$.

3. We have

$$\begin{aligned} \mathrm{epi}\, f &= \{(x,t) : t \geq f_1(x), t \geq f_2(x), x \in (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)\} \\ &= \mathrm{epi}(f_1) \cap \mathrm{epi}(f_2). \end{aligned}$$

Operations with convex functions

Note that a property similar to the second property for the closed convex functions above (for $f(x) = f_1(x) + f_2(x)$, $\mathrm{dom}\, f = (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2)$) does not hold for convex sets.

Example:

- $Q_1 = \{(x,y) : y \geq \frac{1}{x}, x > 0\}$;
- $Q_2 = \{(x,y) : y = 0, x \leq 0\}$;
- $Q_1 + Q_2 = \{(x,y) : y > 0\}$.

Here Q_1, Q_2 are closed and convex, however $Q_1 + Q_2$ is open and convex.

Operations with convex functions

Theorem (3.1.6)

Let function $\phi(y), y \in \mathbb{R}^m$, be convex and closed. Then for a linear operator

$$\mathcal{A}(x) = Ax + b : \mathbb{R}^n \to \mathbb{R}^m,$$

the function $f(x) = \phi(\mathcal{A}(x))$ is a closed convex function with the domain

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n : \mathcal{A}(x) \in \mathrm{dom}\, \phi\}.$$

Operations with convex functions

Theorem (3.1.7)

Let Δ be some set and

$$f(x) = \sup_{y\in\Delta} \phi(y,x),$$

where for any fixed $y \in \Delta$ the function $\phi(y,x)$ is closed and convex in x. Then $f(x)$ is a closed and convex function with the domain

$$\mathrm{dom}(f) = \{x \in \bigcap_{y\in\Delta} \mathrm{dom}\, \phi(y,\cdot) | \exists\gamma : \phi(y,x) \leq \gamma \forall y \in \Delta\}.$$

Proof: Denote by
$S = \{x \in \bigcap_{y\in\Delta} \mathrm{dom}\, \phi(y,\cdot) | \exists\gamma : \phi(y,x) \leq \gamma \forall y \in \Delta\}$.

- If $x \in S$ then $f(x) < \infty$ and $x \in \mathrm{dom}\, f$.
  If $x \notin S$, then there exists a sequence $\{y_k\}$ such that $\phi(y_k, x) \to \infty, k \to \infty \Rightarrow x \notin \mathrm{dom}\, f$.

Operations with convex functions

- $(x, t) \in \mathrm{epi}\ f$ if and only if for all $y \in \Delta$ we have

  $$x \in \mathrm{dom}\ \phi(y, \cdot),\ t \geq \phi(y, x).$$

  Thus,

  $$\mathrm{epi}\ f = \bigcap_{y \in \Delta} \mathrm{epi}\ \phi(y, \cdot),$$

  and f is convex and closed since $\mathrm{epi}\ \phi(y, \cdot)$ is convex and closed. $\square$