ISEN 629: Engineering Optimization

Lecture 12

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Operations with convex functions

Examples:

1. $f(x) = \max_{1 \le i \le n} \{x^{(i)}\}$ is closed and convex.
2. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and Δ be a set in $\mathbb{R}^m_+$. Consider the function
$$f(x) = \sup_{\lambda \in \Delta} \sum_{i=1}^{m} \lambda^{(i)} f_i(x),$$
where f_i are closed and convex. Then f is closed and convex.
3. Let Q be a convex set. Consider the function
$$\psi_Q(x) = \sup_{g \in Q} g^T x.$$
Then $\psi_Q(x)$ is closed and convex.
Function $\psi_Q(x)$ is called the support function of the set Q. It is homogeneous of degree one:
$$\psi_Q(tx) = t\psi_Q(x), x \in \mathrm{dom}\ Q, t \ge 0.$$

A closed convex function that is not continuous

4. Let $Q \subseteq \mathbb{R}^n$. Consider the function $\psi(g, \gamma) = \sup_{y \in Q} \phi(y, g, \gamma)$,

where
$$\phi(y, g, \gamma) = g^T y - \frac{\gamma}{2}\|y\|^2.$$

- $\psi(g, \gamma)$ is closed and convex in (g, γ).
- If Q is bounded then $\mathrm{dom}\ \psi = \mathbb{R}^{n+1}$.
- If $Q = \mathbb{R}^n$, then $\mathrm{dom}\ \psi$ contains only points with $\gamma \ge 0$.
Indeed, if $\gamma < 0$ then for any $g \ne 0$ we can build a sequence $y_k = kg$ such that $\phi(y_k, g, \gamma) \to \infty, k \to \infty$.
  - If $\gamma = 0$, then $g = 0$ (otherwise, $\phi(y, g, 0)$ is unbounded).
  - If $\gamma > 0$ then $y^*(g, \gamma) = \frac{1}{\gamma} g$ is the maximizer of $\phi(y, g, \gamma)$ with respect to y, and $\psi(g, \gamma) = \frac{\|g\|^2}{2\gamma}$. Thus,

$$\psi(g, \gamma) = \begin{cases} 0, & \text{if } g = 0, \gamma = 0, \\ \frac{\|g\|^2}{2\gamma}, & \text{if } \gamma > 0, \end{cases}$$

with $\mathrm{dom}\ \psi = (\mathbb{R}^n \times \{\gamma > 0\}) \cup (0, 0)$. This is a convex set, neither open, nor closed. ψ is a closed convex function, which is not continuous in $(0, 0)$, since $\psi(\sqrt{\gamma} g, \gamma) = \frac{1}{2}\|g\|^2, \gamma \ne 0$.

Convexity and continuity

Lemma (3.1.2)

Let f be convex and $x_0 \in \mathrm{int}(\mathrm{dom}\ f)$. Then f is locally upper bounded at x_0.

Proof: Choose $\epsilon > 0$ such that $x_0 \pm \epsilon e_i \in \mathrm{int}(\mathrm{dom}\ f),\ i = 1, \dots, n$, and denote by
$$\Delta = \mathrm{Conv}\{x_0 \pm \epsilon e_i, i = 1, \dots, n\}.$$

1. We can show that $\Delta \supset B_{\bar{\epsilon}}(x_0)$ with $\bar{\epsilon} = \epsilon/\sqrt{n}$.
2. Thus,
$$M \equiv \max_{x \in B_{\bar{\epsilon}}(x_0)} f(x) \le \max_{x \in \Delta} f(x) \le \max_{1 \le i \le n} f(x_0 \pm \epsilon e_i). \quad \square$$

Convexity and continuity

Theorem (3.1.8)

Let f be convex and $x_0 \in \mathrm{int}(\mathrm{dom}\, f)$. Then f is locally Lipschitz continuous at x_0.

Proof: Fix ϵ such that $B_\epsilon(x_0) \subseteq \mathrm{dom}\, f$ and M such that $\sup\{f(x) : x \in B_\epsilon(x_0)\} \leq M$. Let $x_0 \neq y \in B_\epsilon(x_0)$. Denote by

$$\alpha = \frac{1}{\epsilon}\|y - x_0\|, \; z = x_0 + \frac{1}{\alpha}(y - x_0).$$

Then $\alpha \leq 1$ and $y = \alpha z + (1-\alpha)x_0$. Due to convexity and since $\|z - x_0\| = \frac{1}{\alpha}\|y - x_0\| = \epsilon$:

$$\begin{aligned} f(y) &\leq \alpha f(z) + (1-\alpha)f(x_0) \leq f(x_0) + \alpha(M - f(x_0)) \\ &= f(x_0) + \tfrac{M - f(x_0)}{\epsilon}\|y - x_0\|. \end{aligned}$$

To finish the proof, we need to show that

$$f(y) \geq f(x_0) - \frac{M - f(x_0)}{\epsilon}\|y - x_0\|.$$

Convexity and continuity

Choose $u = x_0 + \frac{1}{\alpha}(x_0 - y)$. Then $\|u - x_0\| = \epsilon$ and $y = x_0 + \alpha(x_0 - u)$.

Recall a property of convex functions:

f is convex if and only if for any $x, y \in \mathrm{dom}\, f$ and $\beta \geq 0$ such that $y + \beta(y - x) \in \mathrm{dom}\, f$, we have

$$f(y + \beta(y - x)) \geq f(y) + \beta(f(y) - f(x))$$

Using this inequality with $y = x_0, x = u, \beta = \alpha$, we obtain:

$$\begin{aligned} f(y) &\geq f(x_0) + \alpha(f(x_0) - f(u)) \geq f(x_0) - \alpha(M - f(x_0)) \\ &= f(x_0) - \tfrac{M - f(x_0)}{\epsilon}\|y - x_0\|. \qquad \square \end{aligned}$$

Convexity and differentiability

Definition

We call f differentiable in a direction p at point $x \in \mathrm{dom}\, f$ if the following limit exists:

$$f'(x; p) = \frac{\partial f(x)}{\partial p} = \lim_{\alpha \to 0+} \frac{1}{\alpha}[f(x + \alpha p) - f(x)].$$

The value $f'(x; p)$ is called the directional derivative of f at x in the direction p.

Theorem (3.1.9)

Convex function f is differentiable in any direction at any interior point of $\mathrm{dom}\, f$.

Proof: Let $x \in \mathrm{int}(\mathrm{dom}\, f)$ and denote by

$$\phi(\alpha) = \frac{1}{\alpha}[f(x + \alpha p) - f(x)], \; \alpha > 0.$$

To prove the statement it suffices to show that $\phi(\alpha)$ is decreasing and bounded from below.

Convexity and differentiability

1. Let $\epsilon > 0$ be small enough to have $x + \epsilon p \in \mathrm{dom}\, f$. Then for $\alpha \in (0, \epsilon]$ and $\beta \in (0, 1]$:

   $$f(x + \alpha\beta p) = f((1-\beta)x + \beta(x + \alpha p)) \leq (1-\beta)f(x) + \beta f(x + \alpha p).$$

   $$\phi(\alpha\beta) = \frac{1}{\alpha\beta}[f(x + \alpha\beta p) - f(x)] \leq \frac{1}{\alpha}[f(x + \alpha p) - f(x)] = \phi(\alpha).$$

   This means that $\phi(\alpha)$ decreases as $\alpha \to 0+$.
2. Let us choose $\gamma > 0$ small enough to have $x - \gamma p \in \mathrm{dom}\, f$. Then using the inequality

   $$f(\bar{y} + \beta(\bar{y} - \bar{x})) \geq f(\bar{y}) + \beta(f(\bar{y}) - f(\bar{x}))$$

   with $\bar{y} = x, \bar{x} = x - \gamma p, \beta = \frac{\alpha}{\gamma}$, we obtain

   $$f(x + \alpha p) \geq f(x) + \frac{\alpha}{\gamma}[f(x) - f(x - \gamma p)] \Leftrightarrow \phi(\alpha) \geq \frac{1}{\gamma}[f(x) - f(x - \gamma p)].$$

Convexity and differentiability

Lemma (3.1.3)

Let f be a convex function and $x \in \operatorname{int}(\operatorname{dom} f)$. Then

1. *$f'(x;p)$ is a homogeneous function of p of degree 1;*
2. *$f'(x;p)$ is a convex function of p;*
3. *for any $y \in \operatorname{dom} f$ we have*

$$f(y) \geq f(x) + f'(x; y-x).$$

Proof:

1. For $p \in \mathbb{R}^n$ and $\tau > 0$ we have

$$\begin{aligned} f'(x, \tau p) &= \lim_{\alpha \to 0+} \tfrac{1}{\alpha}[f(x+\tau\alpha p) - f(x)] \\ &= \tau \lim_{\beta \to 0+} \tfrac{1}{\beta}[f(x+\beta p) - f(x)] = \tau f'(x_0; p). \end{aligned}$$

Convexity and differentiability

2. For any $p_1, p_2 \in \mathbb{R}^n$ and $\beta \in [0,1]$ we have $f'(x; \beta p_1 + (1-\beta)p_2)$

$$\begin{aligned} &= \lim_{\alpha \to 0+} \tfrac{1}{\alpha}[f(x + \alpha(\beta p_1 + (1-\beta)p_2)) - f(x)] \\ &\leq \lim_{\alpha \to 0+} \tfrac{1}{\alpha}(\beta[f(x+\alpha p_1) - f(x)] + (1-\beta)[f(x+\alpha p_2) - f(x)]) \\ &= \beta f'(x; p_1) + (1-\beta) f'(x; p_2). \end{aligned}$$

3. Let $\alpha \in (0,1], y \in \operatorname{dom} f$, and $y_\alpha = x + \alpha(y-x)$. Then using the inequality

$$f(\bar{y} + \beta(\bar{y} - \bar{x})) \geq f(\bar{y}) + \beta(f(\bar{y}) - f(\bar{x}))$$

with $\bar{y} = y_\alpha, \bar{x} = x, \beta = \frac{1}{\alpha}(1-\alpha)$, we obtain

$$f(y) = f(y_\alpha + \frac{1}{\alpha}(1-\alpha)(y_\alpha - x)) \geq f(y_\alpha) + \frac{1}{\alpha}(1-\alpha)[f(y_\alpha) - f(x)].$$

Taking the limit in $\alpha \to 0+$, we obtain

$$f(y) \geq f(x) + f'(x; y-x). \qquad \square$$

Separation theorems

Definition (Supporting hyperplane; separating hyperplane)

Let Q be a convex set and

$$\mathcal{H}(g,\gamma) = \{x \in \mathbb{R}^n : g^T x = \gamma\}, g \neq 0,$$

is a hyperplane.

- We say that the hyperplane $\mathcal{H}(g,\gamma)$ is a supporting hyperplane of Q if $\mathcal{H}(g,\gamma) \cap Q \neq \emptyset$ and $Q \subseteq \mathcal{H}_+(g,\gamma)$ or $C \subseteq \mathcal{H}_-(g,\gamma)$.
- We say that the hyperplane $\mathcal{H}(g,\gamma)$ separates a point x_0 from Q if

$$g^T x \leq \gamma \leq g^T x_0 \; [\text{or } g^T x_0 \leq \gamma \leq g^T x]$$

  for all $x \in Q$. If one of the inequalities is strict, we call the separation strict.

Separation theorems

Definition

Let Q be a closed set and $x_0 \in \mathbb{R}^n$. Then the projection $\pi_Q(x_0)$ of point x_0 onto the set Q is given by

$$\pi_Q(x_0) = \arg\min\{\|x - x_0\| : x \in Q\}.$$

Theorem (3.1.10)

If Q is a convex set, then there exists a unique projection $\pi_Q(x_0)$.

Proof:

$$\pi_Q(x_0) = \arg\min\{\phi(x) : x \in Q\},$$

where $\phi(x) = \frac{1}{2}\|x - x_0\|^2$ belongs to $\mathcal{S}^{1,1}_{1,1}(\mathbb{R}^n)$. $\square$

Separation theorems

Lemma (3.1.4)

Let Q be a closed convex set and $x_0 \notin Q$. Then for any $x \in Q$:

$$(\pi_Q(x_0) - x_0)^T (x - \pi_Q(x_0)) \geq 0.$$

Proof: Since $\pi_Q(x_0) = \arg\min\{\phi(x) : x \in Q\}$, where $\phi(x) = \frac{1}{2}\|x - x_0\|^2$, we have

$$\phi'(\pi_Q(x_0))^T (x - \pi_Q(x_0)) \geq 0 \ \forall x \in Q.$$

Noting that $\phi'(x) = x - x_0$ and thus $\phi'(\pi_Q(x_0)) = \pi_Q(x_0) - x_0$, we obtain the inequality of the lemma. □

Separation theorems

Lemma (3.1.5)

For any $x \in Q$ we have

$$\|x - \pi_Q(x_0)\|^2 + \|\pi_Q(x_0) - x_0\|^2 \leq \|x - x_0\|^2.$$

Proof:

$\|x - \pi_Q(x_0)\|^2 - \|x - x_0\|^2$

$$\begin{aligned}
&= -2x^T \pi_Q(x_0) + \|\pi_Q(x_0)\|^2 + 2x^T x_0 - \|x_0\|^2 \\
&= (x_0 - \pi_Q(x_0))^T (2x - \pi_Q(x_0) - x_0) \\
&= (x_0 - \pi_Q(x_0))^T (2x - 2\pi_Q(x_0) + \pi_Q(x_0) - x_0) \\
&= -2(\pi_Q(x_0) - x_0)^T (x - \pi_Q(x_0)) - \|\pi_Q(x_0) - x_0\|^2 \\
&\leq -\|x_0 - \pi_Q(x_0)\|^2. \quad \square
\end{aligned}$$