ISEN 629: Engineering Optimization

Lecture 13

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Separation theorems

Theorem (3.1.11)

Let Q be a closed convex set and $x_0 \notin Q$. Then there exists a hyperplane $\mathcal{H}(g, \gamma)$, which strictly separates x_0 from Q.

Proof.
Take $g = x_0 - \pi_Q(x_0) \neq 0,\ \gamma = (x_0 - \pi_Q(x_0))^T \pi_Q(x_0)$. Then, since for any $x \in Q$,

$$(\pi_Q(x_0) - x_0)^T (x - \pi_Q(x_0)) \geq 0,$$

for any $x \in Q$ we have

$$\begin{aligned} (x_0 - \pi_Q(x_0))^T x &\leq (x_0 - \pi_Q(x_0))^T \pi_Q(x_0) \\ &= (x_0 - \pi_Q(x_0))^T x_0 - \|x_0 - \pi_Q(x_0)\|^2 \\ &< (x_0 - \pi_Q(x_0))^T x_0, \end{aligned}$$

i.e., for any $x \in Q$: $g^T x \leq \gamma < g^T x_0$. □

Separation theorems

Corollary

Let Q_1 and Q_2 be two closed convex sets and $\psi_Q(g)$ be the support function of set Q ($\psi_Q(g) = \sup\{g^T x : x \in Q\}$).

1. *If for all $g \in \mathrm{dom}\,\psi_{Q_2}$ we have $\psi_{Q_1}(g) \leq \psi_{Q_2}(g)$, then $Q_1 \subseteq Q_2$.*
2. *If $\mathrm{dom}\,\psi_{Q_1} = \mathrm{dom}\,\psi_{Q_2}$ and for all $g \in \mathrm{dom}\,\psi_{Q_1}$ we have $\psi_{Q_1}(g) = \psi_{Q_2}(g)$, then $Q_1 = Q_2$.*

Proof.

1. Assume that there exists $x_0 \in Q_1 \setminus Q_2$. Then there exists a direction g such that for any $x \in Q_2$

$$g^T x_0 > \gamma > g^T x$$

   Thus, $g \in \mathrm{dom}\,\psi_{Q_2}$ and $\psi_{Q_1}(g) > \psi_{Q_2}(g)$ - a contradiction.
2. Due to the first statement, $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$. □

Separation theorems

Theorem (3.1.12)

Let Q be a closed convex set and x_0 belong to the boundary of set Q. Then there exists a supporting hyperplane $\mathcal{H}(g, \gamma)$ for Q passing through x_0.

Proof. Consider a sequence $\{y_k\}$ such that $y_k \notin Q$ and $y_k \to x_0, k \to \infty$. Denote by

$$g_k = \frac{y_k - \pi_Q(y_k)}{\|y_k - \pi_Q(y_k)\|}, \gamma_k = g_k^T \pi_Q(y_k).$$

Then for all $x \in Q$ we have

$$g_k^T x \leq \gamma_k < g_k^T y_k.$$

Note that $\|g_k\| = 1$ and the sequence $\{\gamma_k\}$ is bounded:

$$\begin{aligned} |\gamma_k| &= |g_k^T(\pi_Q(y_k) - x_0) + g_k^T x_0| \\ &\leq \|\pi_Q(y_k) - x_0\| + \|x_0\| \leq \|y_k - x_0\| + \|x_0\|. \end{aligned}$$

Separation theorems

Since a bounded sequence always has a limit point, without loss of generality we can assume that

$$g^* = \lim_{k\to\infty} g_k \text{ and } \gamma^* = \lim_{k\to\infty} \gamma_k.$$

Taking the limit in

$$g_k^T x \le \gamma_k < g_k^T y_k,$$

we obtain:

$${g^*}^T x \le \gamma^* \le {g^*}^T x_0 \; \forall x \in Q,$$

so $\gamma^* = {g^*}^T x_0$. □

Subgradients

Definition

For a convex function f, a vector g is called a subgradient of f at point $x_0 \in \mathrm{dom}\, f$ if for any $x \in \mathrm{dom}\, f$ we have

$$f(x) \ge f(x_0) + g^T(x - x_0).$$

The set of subgradients of f at x_0 is denoted by $\partial f(x_0)$ and is called the subdifferential of f at x_0.

Example

Let $f(x) = |x|, x \in \mathbb{R},\ x_0 = 0$. Then a subgradient of f at x_0 is any g satisfying

$$|x| \ge gx$$

for all $x \in \mathbb{R}$. This inequality is satisfied for all $x \in \mathbb{R}$ if and only if $g \in [-1, 1]$.

Subgradients

- Note that the set $\partial f(x_0)$ is defined by the set of linear constraints:

  $$\partial f(x_0) = \{g \in \mathbb{R}^n : f(x) \ge f(x_0) + g^T(x - x_0),\ x \in \mathrm{dom}\, f\}.$$

  Therefore, the subdifferential is a closed convex set.
- Next, we will show that subdifferentiability implies convexity.

Subgradients

Lemma (3.1.6)

If $\partial f(x)$ is nonempty for any $x \in \mathrm{dom}\, f$, then f is convex.

Proof.

Consider $z = \alpha y + (1-\alpha)x$, where $x, y \in \mathrm{dom}\, f$, $\alpha \in [0, 1]$. Let $g \in \partial f(z)$. Then

$$\begin{aligned} f(y) &\ge f(z) + g^T(y - z) = f(z) + (1-\alpha)g^T(y - x) \\ f(x) &\ge f(z) + g^T(x - z) = f(z) - \alpha g^T(y - x) \end{aligned}$$

Multiplying the inequalities by α and $(1-\alpha)$, respectively, we obtain

$$\alpha f(y) + (1-\alpha) f(x) \ge f(z).$$

□

Subgradients

Theorem (3.1.13)

Let f be closed and convex and $x_0 \in \mathrm{int}(\mathrm{dom}\, f)$. Then $\partial f(x_0)$ is a nonempty bounded set.

Proof.

$(x_0, f(x_0))$ belongs to the boundary of $\mathrm{epi}(f)$. Hence, there exists a supporting hyperplane for epi(f) at $(x_0, f(x_0))$:

$$d^T x - \alpha y \leq d^T x_0 - \alpha f(x_0) \quad \forall (x, y) \in \mathrm{epi}(f), \qquad (1)$$

where d and α are selected so that $\|d\|^2 + \alpha^2 = 1$.
Since for all $(x_0, y) \in \mathrm{epi}(f)$ we have $y \geq f(x_0)$, using $x = x_0$ in (1) we conclude that $\alpha \geq 0$.

The rest of the proof consists in showing that:

1. $\alpha > 0$;
2. $g = d/\alpha \in \partial f(x_0)$;
3. for any $g \in \partial f(x_0)$ we have $\|g\| \leq M$.

Subgradients

1. Since a convex function is bounded in the interior of its domain, there exist $\epsilon > 0, M > 0$ such that $B_\epsilon(x_0) \subseteq \mathrm{dom}\, f$ and
$$f(x) - f(x_0) \leq M\|x - x_0\| \quad \forall x \in B_\epsilon(x_0).$$
Therefore, using (1) for any $x \in B_\epsilon(x_0) \subseteq \mathrm{dom}\, f$ we have
$$d^T (x - x_0) \leq \alpha(f(x) - f(x_0)) \leq \alpha M \|x - x_0\|$$
Choosing $x = x_0 + \epsilon d$ we have $\|d\|^2 \leq M\alpha\|d\|$, and since $\|d\|^2 = 1 - \alpha^2$, we obtain $\alpha \geq \frac{1}{\sqrt{1+M^2}} > 0$.
2. Thus, with $g = d/\alpha$, (1) is equivalent to
$$f(x) \geq f(x_0) + g^T (x - x_0) \quad \Rightarrow \quad g \in \partial f(x_0).$$
3. If $g \in \partial f(x_0)$, then for $x = x_0 + \epsilon g/\|g\| \in B_\epsilon(x_0)$ we have:
$$\epsilon\|g\| = g^T (x - x_0) \leq f(x) - f(x_0) \leq M\|x - x_0\| = M\epsilon. \quad \square$$

Subgradients

Example

Consider the function $f(x) = -\sqrt{x}$ with the domain $\{x \in \mathbb{R} : x \geq 0\}$

- $f(x)$ is convex and closed;
- $\partial f(0) = \emptyset$.

Note that $x_0 = 0 \notin \mathrm{int}(\mathrm{dom}\, f)$, so the condition of the theorem is violated.