ISEN 629: Engineering Optimization

Lecture 18

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Constrained minimization

Consider the problem

$$\begin{array}{rl} & \min\ f(x), \\ \text{s.t.} & f_j(x) \le 0,\ j = 1, \dots, m, \\ & x \in Q, \end{array}$$

where Q is a compact convex set and $f(x), f_j(x)$ are Lipschitz continuous on Q.
Denoting by $\bar{f}(x) = \max\limits_{1 \le j \le m} f_j(x)$, we obtain the equivalent problem

$$\begin{array}{rl} & \min\ f(x), \\ \text{s.t.} & \bar{f}(x) \le 0, \\ & x \in Q, \end{array} \qquad (*)$$

Then $f(x), \bar{f}(x)$ are convex and Lipschitz continuous on Q.

Constrained minimization

Let us introduce the *parametric function*:

$$f(t; x) = \max\{f(x) - t, \bar{f}(x)\}$$

and denote by

$$f^*(t) = \min_{x \in Q} f(t; x). \qquad (1)$$

Note that $f^*(t)$ is nonincreasing in t.

Lemma

Let x^* be the optimal solution of problem $(*)$ with $t^* = f(x^*)$ being the optimal objective value. Then

$$\begin{array}{rcl} f^*(t) & \le & 0 \text{ for all } t \ge t^*, \\ f^*(t) & > & 0 \text{ for all } t < t^*. \end{array}$$

Thus, t^* is the smallest root of function $f^*(t)$ and it corresponds to the optimal value of $(*)$.

Constrained minimization

To solve $(*)$, we will use models of both $f(x)$ and $\bar{f}(x)$ (which will be used to introduce a model function for $f(t, x)$).
For a sequence $X = \{x_k : k \ge 0\}$, denote by

$$\begin{array}{rcl} \hat{f}_k(X; x) & = & \max\limits_{0 \le j \le k} [f(x_j) + g(x_j)^T (x - x_j)] \le f(x), \\ \check{f}_k(X; x) & = & \max\limits_{0 \le j \le k} [\bar{f}(x_j) + \bar{g}(x_j)^T (x - x_j)] \le \bar{f}(x), \end{array}$$

where $g(x_j) \in \partial f(x_j)$ and $\bar{g}(x_j) \in \partial \bar{f}(x_j)$.

Constrained minimization

The models of $f(x)$ and $\bar{f}(x)$ can be used to introduce the model for $f(t;x)$:

$$\begin{aligned} f_k(X;t,x) &= \max\{\hat{f}_k(X;x) - t, \check{f}_k(X,x)\} \leq f(t;x), \\ \hat{f}_k^*(X;t) &= \min_{x \in Q} f_k(X;t,x) \leq f^*(t). \end{aligned}$$

Then $\hat{f}_k^*(X;t)$ is nonincreasing in t and its smallest root $t_k^*(X)$ does not exceed t^*.

Lemma (3.3.4)

$$t_k^*(X) = \min\{\hat{f}_k(X;x) : \check{f}_k(X;x) \leq 0, x \in Q\}.$$

Denote by

$$f_k^*(X;t) = \min_{0 \leq j \leq k} f_k(X;t,x_j)$$

the record value of our parametric model.

Constrained level method

0. Choose $x_0 \in Q, t_0 < t^*, \kappa \in (0, 1/2)$ and $\epsilon > 0$.
1. k-th iteration ($k \geq 0$):
   (a) Keep generating sequence $X = \{x_j : j \geq 0\}$ by the level method applied to function $f(t_k;x)$. If the internal termination criterion

   $$\hat{f}_j^*(X;t_k) \geq (1-\kappa) f_j^*(X;t_k)^\dagger$$

   holds, then stop the internal process and set $j(k) = j$.
   Global stop: $f_j^*(X;t_k) \leq \epsilon$.
   (b) Set $t_{k+1} = t_{j(k)}^*(X)$.

†same as $f_j^*(X;t_k) - \hat{f}_j^*(X;t_k) \leq \delta$ with $\delta = \kappa f_j^*(X;t_k)$

Constrained level method

Let the global stop condition be satisfied:

$$f_j^*(X;t_k) \leq \epsilon.$$

Then there exists j^* such that

$$f(t_k, x_{j^*}) = f_j^*(X;t_k) \leq \epsilon.$$

Therefore, we have

$$f(t_k, x_{j^*}) = \max\{f(x_{j^*}) - t_k; \bar{f}(x_{j^*})\} \leq \epsilon.$$

Since $t_k \leq t^*$, we conclude that

$$\begin{aligned} f(x_{j^*}) &\leq t^* + \epsilon, \\ \bar{f}(x_{j^*}) &\leq \epsilon. \end{aligned}$$

Constrained level method: complexity

To derive the analytical complexity bound for the constrained level method, we need to estimate

1. the complexity of the master process (i.e., the maximum number of iterations indexed by k);
2. the complexity of step 1(a).

Constrained level method: master process complexity

Lemma (3.3.7)

For all $k \geq 0$ we have

$$f^*_{j(k)}(X; t_k) \leq \frac{t_0 - t^*}{1-\kappa}\left[\frac{1}{2(1-\kappa)}\right]^k.$$

> *Thus, the algorithm terminates in at most*
>
> $$N(\epsilon) = \frac{1}{\ln[2(1-\kappa)]}\ln\frac{t_0 - t^*}{(1-\kappa)\epsilon}$$
>
> *full iterations of the master process.*

[full iterations are iterations terminated by the internal termination criterion]

Constrained level method: internal process complexity

Denote by $M_f = \max\{\|g\| : g \in \partial f(x) \cup \partial \bar{f}(x), x \in Q\}$.

We need to analyze two cases: (1) Termination by internal criterion (full step) and (2) Termination by global criterion (last step).

1. Full step. The internal process is terminated by the rule

$$f^*_j(X; t_k) - \hat{f}^*_j(X; t_k) \leq \kappa f^*_j(X; t_k),$$

where $f^*_j(X; t_k) \geq \epsilon$.

Recall Theorem (3.3.1): Let $\mathrm{diam}(Q) = D$. Then the level method needs at most $N = \left\lfloor \frac{M_f^2 D^2}{\epsilon^2\alpha(1-\alpha)^2(2-\alpha)} \right\rfloor + 1$ iterations to guarantee $f^*_k - f^* \leq \epsilon$.

> *Thus,*
>
> $$j(k) - j(k-1) \leq \frac{M_f^2 D^2}{\kappa^2\epsilon^2\alpha(1-\alpha)^2(2-\alpha)}.$$
>
> *for any full iteration of the master process.*

Constrained level method: internal process complexity

2. Last step. The internal process was terminated by the global stop rule:

$$f^*_j(X; t_k) \leq \epsilon.$$

Since the internal termination criterion was not satisfied, we have

$$f^*_{j-1}(X; t_k) - \hat{f}^*_{j-1}(X; t_k) \geq \kappa f^*_{j-1}(X; t_k) \geq \kappa\epsilon.$$

Thus, the number of iterations of the level method at the last step does not exceed

$$\frac{M_f^2 D^2}{\kappa^2\epsilon^2\alpha(1-\alpha)^2(2-\alpha)}.$$

Constrained level method: total complexity

Thus, the total complexity of the constrained level method is

$$(N(\epsilon)+1)\frac{M_f^2 D^2}{\kappa^2\epsilon^2\alpha(1-\alpha)^2(2-\alpha)}$$

$$= \frac{M_f^2 D^2}{\kappa^2\epsilon^2\alpha(1-\alpha)^2(2-\alpha)}\left[1 + \frac{1}{\ln[2(1-\kappa)]}\ln\frac{t_0-t^*}{(1-\kappa)\epsilon}\right]$$

$$= \frac{M_f^2 D^2 \ln\frac{2(t_0-t^*)}{\epsilon}}{\kappa^2\epsilon^2\alpha(1-\alpha)^2(2-\alpha)\ln[2(1-\kappa)]}.$$

It can be shown that the following values of parameters are reasonable:

$$\alpha = \kappa = 1 - \frac{1}{\sqrt{2}}.$$

> *The total complexity of the constrained level method is*
>
> $$O\left(\frac{1}{\epsilon^2}\ln\frac{2(t_0-t^*)}{\epsilon}\right).$$

Constrained level method

0. Choose $x_0 \in Q, t_0 < t^*, \kappa \in (0, 1/2)$ and $\epsilon > 0$.
1. k-th iteration ($k \geq 0$):
   (a) Keep generating sequence $X = \{x_j : j \geq 0\}$ by the level method applied to function $f(t_k; x)$. If the internal termination criterion
   $$\hat{f}_j^*(X; t_k) \geq (1 - \kappa) f_j^*(X; t_k)$$
   holds, then stop the internal process and set $j(k) = j$.
   Global stop: $f_j^*(X; t_k) \leq \epsilon$.
   (b) Set $t_{k+1} = t_{j(k)}^*(X)$.

Remaining issues to discuss:

- Computing $\hat{f}_j^*(X; t_k)$ - reduces to LP if Q is a polytope [previous lecture].
- Finding the root $t_{j(k)}^*(X)$.

Constrained level method: finding $t_{j(k)}^*(X)$

Recall Lemma (3.3.4): $t_k^*(X) = \min\{\hat{f}_k(X; x) : \check{f}_k(X; x) \leq 0, x \in Q\}$.

Thus, finding $t_{j(k)}^*(X)$ is equivalent to solving the problem

$$\min\{\hat{f}_k(X; x) : \check{f}_k(X; x) \leq 0, x \in Q\},$$

which is equivalent to

$$\begin{array}{ll} & \min\ t, \\ \text{s.t.} & f(x_j) + g(x_j)^T(x - x_j) \leq t,\ j = 0, \ldots, k, \\ & \bar{f}(x_j) + \bar{g}(x_j)^T(x - x_j) \leq 0,\ j = 0, \ldots, k, \\ & x \in Q. \end{array}$$

- If Q is a polytope, this is an LP.
- If Q is more complicated, we can use interior point methods.

Constrained level method

Remark: Since $\bar{f}(x) = \max\limits_{1 \leq i \leq m} f_i(x)$, we can use

$$\check{f}_k(X; x) = \max_{0 \leq j \leq k} \max_{1 \leq i \leq m} [f_i(x_j) + g_i(x_j)^T(x - x_j)] \leq \bar{f}(x),$$

where $g_i(x_j) \in \partial f_i(x_j)$, instead of

$$\check{f}_k(X; x) = \max_{0 \leq j \leq k} [\bar{f}(x_j) + \bar{g}(x_j)^T(x - x_j)] \leq \bar{f}(x).$$

In practice, this *complete* model accelerates the convergence of the process at expense of increased per-iteration complexity.