ISEN 629: Engineering Optimization

Lecture 21

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Standard Newton method

0. Choose $x_0 \in \mathrm{dom}\, f$.
1. $x_{k+1} = x_k - [f''(x_k)]^{-1} f'(x_k),\, k \geq 0$.

Denote by

$$r_*(x_k) = \|x_k - x_f^*\|_{x_f^*}.$$

Theorem (4.1.13)

Let $\lambda_f(x) < 1$. Then

$$\omega(\lambda_f(x)) \leq f(x) - f(x_f^*) \leq \omega_*(\lambda_f(x)),$$

$$\omega'(\lambda_f(x)) \leq \|x - x_f^*\|_x \leq \omega_*'(\lambda_f(x)),$$

$$\omega(r_*(x)) \leq f(x) - f(x_f^*) \leq \omega_*(r_*(x)),$$

where the last inequality holds for $r_*(x) < 1$.

Standard Newton method

Theorem (4.1.14)

Let $x \in \mathrm{dom}\, f$ and $\lambda_f(x) < 1$. Then the point

$$x_+ = x - [f''(x)]^{-1} f'(x)$$

belongs to $\mathrm{dom}\, f$ and

$$\lambda_f(x_+) \leq \left(\frac{\lambda_f(x)}{1 - \lambda_f(x)} \right)^2.$$

Thus, to guarantee that $\lambda_f(x_+) < \lambda_f(x)$, we need to have

$$\lambda_f(x) < \bar{\lambda} = \frac{3 - \sqrt{5}}{2} = 0.3819...,$$

where $\bar{\lambda}$ is the root of the equation $\frac{\lambda}{(1-\lambda)^2} = 1$.

Solution strategy

- First stage: $\lambda_f(x_k) \geq \beta$, where $\beta \in (0, \bar{\lambda})$. Apply the damped Newton method. At each iteration we have

  $$f(x_{k+1}) \leq f(x_k) - \omega(\beta).$$

  The number of steps N for this stage is bounded by

  $$N \leq \frac{1}{\omega(\beta)} [f(x_0) - f(x_f^*)].$$

- Second stage: $\lambda_f(x_k) \leq \beta$. Apply the standard Newton method. We have the quadratic rate of convergence.

Even though the damped Newton method also has quadratic rate of convergence, the proposed strategy gives better complexity bound.

Self-concordant barriers: motivation

- Denote by $\mathrm{Dom}\, f = \mathrm{cl}(\mathrm{dom}\, f)$
- By standard constrained minimization problem we will mean a problem in the form
$$\min_{x \in Q} c^T x, \qquad (*)$$
where Q is a closed convex set. We also assume that we know a self-concordant function f such that $\mathrm{Dom}\, f = Q$.
- Consider a parametric barrier function
$$f(t;x) = tc^T x + f(x), \quad t \geq 0.$$
- Note that $f(t;x)$ is self-concordant in x.
- Denote by $x^*(t) = \arg \min_{x \in \mathrm{dom}\, f} f(t;x)$.
- $x^*(t)$ is called the central path of the problem $(*)$.
- Can we expect that $x^*(t) \to x^*$ as $t \to \infty$?

Self-concordant barriers: motivation

Recall the general scheme of the barrier function method:

Barrier function method

0. Choose $x_0 \in \mathrm{int} Q$. Choose a sequence of penalty coefficients: $0 < t_k < t_{k+1}$ and $t_k \to \infty$.
1. k^{th} iteration ($k \geq 0$):
Find a point $x_{k+1} = \arg\min_{x \in Q} \{f_0(x) + \frac{1}{t_k} F(x)\}$ using x_k as a starting point.

Denote by $\Psi_k(x) = f_0(x) + \frac{1}{t_k} F(x)$, $\Psi_k^* = \min_{x \in Q} \Psi_k(x)$.

Theorem (1.3.2)

Let barrier $F(x)$ be bounded from below on Q. Then $\lim_{k\to\infty} \Psi_k^* = f^*$, where f^* is the optimal value of the considered problem.

Self-concordant barriers: motivation

The region of quadratic convergence of the standard Newton method applied to minimization of $f(t;x)$ is given by

$$\lambda_{f(t;\cdot)}(x) \leq \beta < \bar{\lambda} = \frac{3 - \sqrt{5}}{2}.$$

Assume that we know $x = x^*(t)$ for some $t > 0$. Then FONC yield the following central path equation:

$$tc + f'(x) = 0.$$

We want to increase t while keeping x within the region of quadratic convergence of the Newton method for $f(t + \Delta; \cdot)$:

$$t_+ = t + \Delta, \ \Delta > 0;$$

$$\lambda_{f(t+\Delta;\cdot)}(x) \leq \beta < \bar{\lambda}.$$

Self-concordant barriers: motivation

Note that the increase in t does not change the Hessian:

$$f''(t + \Delta; x) = f''(t;x).$$

Also, from the central path equation we have $tc + f'(x) = 0$. Therefore,

$$\lambda_{f(t+\Delta;\cdot)}(x) \leq \beta$$

is equivalent to

$$\begin{aligned} \lambda_{f(t+\Delta;\cdot)}(x) &= \|f'(t+\Delta;x)\|_x = \|tc + \Delta c + f'(x)\|_x \\ &= \Delta \|c\|_x = \tfrac{\Delta}{t} \|f'(x)\|_x \leq \beta. \end{aligned}$$

If we increase t at a linear rate, then the last inequality requires that

$$\lambda_f^2(x) = \|f'(x)\|_x^2 \equiv f'(x)^T f''(x)^{-1} f'(x)$$

is uniformly bounded on $\mathrm{dom}\, f$. This is guaranteed by the following definition of self-concordant barriers...

Self-concordant barriers: definition

Definition

A standard self-concordant function $F(x)$ is called ν-self-concordant barrier for set $\mathrm{Dom}\ F$, if

$$\sup_{u \in \mathbb{R}^n} \left\{ 2F'(x)^T u - u^T F''(x) u \right\} \le \nu$$

for all $x \in \mathrm{dom}\ F$. The value ν is called the parameter of the barrier.

If we assume that $F''(x)$ is nondegenerate, then the above inequality is equivalent to

$$\lambda_F^2(x) = \|F'(x)\|_x^2 \equiv F'(x)^T F''(x)^{-1} F'(x) \le \nu.$$

Equivalent definitions:

1. $(F'(x)^T u)^2 \le \nu (u^T F''(x) u)\ \forall u \in \mathbb{R}^n$.
2. $F''(x) \succeq \frac{1}{\nu} F'(x) F'(x)^T$.

Self-concordant barriers: examples

1. Linear function

$$f(x) = a^T x + b, \mathrm{dom}\ f = \mathbb{R}^n,$$

is not a self-concordant barrier since $f''(x) = 0$.

2. Convex quadratic function

$$f(x) = \frac{1}{2} x^T A x + b^T x + c, \mathrm{dom}\ f = \mathbb{R}^n$$

is not a self-concordant barrier, since

$$\begin{aligned} f'(x)^T f''(x)^{-1} f'(x) &= (Ax+b)^T A^{-1} (Ax+b) \\ &= x^T A x + 2b^T x + bA^{-1} b, \end{aligned}$$

is not bounded from above on $\mathbb{R}^n$.

Self-concordant barriers: examples

3. Logarithmic barrier for a ray:

$$F(x) = -\ln x, \mathrm{dom}\ F = \{x \in \mathbb{R} : x > 0\}.$$

Then $F'(x) = -1/x, F''(x) = 1/x^2 > 0$, and

$$\frac{(F'(x))^2}{F''(x)} = 1.$$

Thus, $F(x)$ is a ν-self-concordant barrier for $\{x \in \mathbb{R} : x > 0\}$ with $\nu = 1$.

Self-concordant barriers: examples

4. Logarithmic barrier for a quadratic region:
Let $A = A^T \succeq 0$. Consider the concave function

$$\phi(x) = -\frac{1}{2} x^T A x + b^T x + c.$$

Let $F(x) = -\ln \phi(x),\ \mathrm{dom}\ F = \{x \in \mathbb{R}^n : \phi(x) > 0\}$. Then

$$\begin{aligned} F'(x)^T u &= -\tfrac{1}{\phi(x)} [b^T u - x^T A u], \\ u^T F''(x) u &= \tfrac{1}{\phi^2(x)} [b^T u - x^T A u]^2 + \tfrac{1}{\phi(x)} u^T A u. \end{aligned}$$

Denote by $\omega_1 = F'(x)^T u$ and $\omega_2 = \frac{1}{\phi(x)} u^T A u$. Then

$$u^T F''(x) u = \omega_1^2 + \omega_2 \ge \omega_1^2,$$

and

$$2F'(x)^T u - u^T F''(x) u \le 2\omega_1 - \omega_1^2 \le 1.$$

Therefore, $F(x)$ is a ν-self-concordant barrier with $\nu = 1$.

Self-concordant barriers: properties

Theorem (4.2.1)

If $F(x)$ is a self-concordant barrier, then $c^T x + F(x)$ is a self-concordant function on $\mathrm{dom}\ F$.

Theorem (4.2.2)

Let F_i be ν_i-self-concordant barriers, $i = 1, 2$. Then the function $F(x) = F_1(x) + F_2(x)$ is a self-concordant barrier for $\mathrm{Dom}\ F = \mathrm{Dom}\ F_1 \cap \mathrm{Dom}\ F_2$ with the parameter $\nu = \nu_1 + \nu_2$.

Theorem (4.2.3)

Let $\mathcal{A}(x) = Ax + b$ be a linear operator, $\mathcal{A}(x) : \mathbb{R}^n \to \mathbb{R}^m$. If $F(y)$ is a ν-self-concordant barrier, then $\Phi(x) = F(\mathcal{A}(x))$ is a ν-self-concordant barrier for the set

$$\mathrm{Dom}\ \Phi = \{x \in \mathbb{R}^n : \mathcal{A}(x) \in \mathrm{Dom}\ F\}.$$

Self-concordant barriers: properties

Theorem (4.2.4)

1. *Let $F(x)$ be a ν-self-concordant barrier. Then for any x and y from $\mathrm{dom}\ F$ we have*

$$F'(x)^T(y - x) < \nu.$$

Moreover, if $F'(x)^T(y - x) \geq 0$ then

$$(F'(y) - F'(x))^T(y - x) \geq \frac{(F'(x)^T(y - x))^2}{\nu - F'(x)^T(y - x)}.$$

2. *A standard self-concordant function $F(x)$ is a ν-self-concordant barrier if and only if*

$$F(y) \geq F(x) - \nu \ln\left(1 - \frac{1}{\nu} F'(x)^T(y - x)\right) \quad \forall x, y \in \mathrm{dom}\ F.$$

Self-concordant barriers: properties

Theorem (4.2.5)

Let $F(x)$ be a ν-self-concordant barrier. Then for any $x \in \mathrm{dom}\ F$ and any $y \in \mathrm{Dom}\ F$ such that

$$F'(x)^T(y - x) \geq 0,$$

we have

$$\|y - x\|_x \leq \nu + 2\sqrt{\nu}.$$

Definition

Let $F(x)$ be a ν-self-concordant barrier for the set $\mathrm{Dom}\ F$. The point

$$x_F^* = \arg\min_{x \in \mathrm{dom}\ F} F(x)$$

is called the analytic center of convex set $\mathrm{Dom}\ F$ generated by the barrier $F(x)$.

Self-concordant barriers: properties

Theorem (4.2.6)

Assume that the analytic center of a ν-self-concordant barrier $F(x)$ exists. Then for any $x \in \mathrm{Dom}\ F$ we have

$$\|x - x_F^*\|_{x_F^*} \leq \nu + 2\sqrt{\nu}.$$

On the other hand, for any $x \in \mathbb{R}^n$ such that $\|x - x_F^*\|_{x_F^*} \leq 1$ we have $x \in \mathrm{Dom}\ F$.

Corollary

If $\mathrm{Dom}\ F$ is bounded, then for any $x \in \mathrm{dom}\ F$ and $v \in \mathbb{R}^n$ we have

$$\|v\|_x^* \leq (\nu + 2\sqrt{\nu})\|v\|_{x_F^*}^*.$$