ISEN 629: Engineering Optimization

Lecture 22

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Path-following scheme

- Consider the standard minimization problem
$$\min_{x\in Q} c^T x, \qquad (*)$$
where Q is a closed convex set with nonempty interior. We also assume that we know a ν-self-concordant barrier $F(x)$ such that $\mathrm{Dom}\ F = Q$.
- We will solve $(*)$ by following the central path
$$x^*(t) = \arg\min_{x\in \mathrm{dom}\ F} f(t;x),$$
where $f(t;x) = tc^T x + F(x)$ and $t \geq 0$.
- Recall that any point of the central path satisfies the equation
$$tc + F'(x^*(t)) = 0.$$
- Since Q is compact, its analytic center $x_F^* = \arg\min_{x\in Q} F(x)$ exists and
$$x^*(0) = x_F^*.$$

Path-following scheme

To follow the central path, we will update the points so that the approximate centering condition is satisfied:

$$\lambda_{f(t;\cdot)}(x) \equiv \|f'(t;x)\|_x^* = \|tc + F'(x)\|_x^* \leq \beta, \qquad (**)$$

where the centering parameter β is sufficiently small ($< (3-\sqrt{5})/2$).

Theorem (4.2.7)

Let c^* denote the optimal value of $(*)$. Then for any $t > 0$ we have
$$c^T x^*(t) - c^* \leq \frac{\nu}{t}.$$
If a point x satisfies the approximate centering condition $(**)$ then
$$c^T x - c^* \leq \frac{1}{t}\left(\nu + \frac{(\beta + \sqrt{\nu})\beta}{1-\beta}\right).$$

Path-following scheme

Assume that $x \in \mathrm{dom}\ F$ and consider the following iterate:

$$\boxed{\begin{aligned} t_+ &= t + \tfrac{\gamma}{\|c\|_x^*}, \\ x_+ &= x - [F''(x)]^{-1}(t_+ c + F'(x)). \end{aligned}}$$

Theorem (4.2.8)

Let x satisfy
$$\|tc + F'(x)\|_x^* \leq \beta$$
with $\beta < \bar{\lambda} = \frac{3-\sqrt{5}}{2}$. Then for any γ such that
$$|\gamma| \leq \frac{\sqrt{\beta}}{1+\sqrt{\beta}} - \beta$$
we have $\|t_+ c + F'(x_+)\|_x^* \leq \beta$.

Path-following scheme

Lemma (4.2.1)

If a point x satisfies the approximate centering condition $(**)$ then

$$\|c\|_x^* \leq \frac{1}{t}(\beta + \sqrt{\nu}).$$

Proof.

We use the following inequalities:

1. $\|f'(t;x)\|_x^* = \|tc + F'(x)\|_x^* \leq \beta$ (approximate centering);
2. $F'(x)^T[F''(x)]^{-1}F'(x) \leq \nu$ (ν-self-concordance).

We have:

$$t\|c\|_x^* = \|f'(t;x) - F'(x)\|_x^* \leq \|f'(t;x)\|_x^* + \|F'(x)\|_x^* \leq \beta + \sqrt{\nu}.$$

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Path-following scheme

$$\begin{aligned} t_+ &= t + \tfrac{\gamma}{\|c\|_x^*}, \\ x_+ &= x - [F''(x)]^{-1}(t_+c + F'(x)). \end{aligned}$$

Let us fix some reasonable parameter values in our scheme:

$$\beta = \frac{1}{9},\ |\gamma| = \frac{\sqrt{\beta}}{1+\sqrt{\beta}} - \beta = \frac{5}{36}.$$

Since $\|c\|_x^* \leq \frac{1}{t}(\beta + \sqrt{\nu})$, if we use $\gamma = 5/36$ we have:

$$t_+ = t + \frac{\gamma}{\|c\|_x^*} \geq \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right) t = \left(1 + \frac{5}{4 + 36\sqrt{\nu}}\right) t,$$

and if $\gamma = -5/36$ we obtain

$$t_+ = t + \frac{\gamma}{\|c\|_x^*} \leq \left(1 - \frac{5}{4 + 36\sqrt{\nu}}\right) t.$$

The general path-following scheme

0. Set $t_0 = 0$. Choose an accuracy $\epsilon > 0$ and $x_0 \in \operatorname{dom} F$ such that
$$\|F'(x_0)\|_{x_0}^* \leq \beta.$$
1. k-th iteration ($k \geq 0$): Set
$$\begin{aligned} t_{k+1} &= t_k + \tfrac{\gamma}{\|c\|_{x_k}^*}, \\ x_{k+1} &= x_k - [F''(x_k)]^{-1}(t_{k+1}c + F'(x_k)). \end{aligned}$$
2. Stop if $\epsilon t_k \geq \nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}$.

The general path-following scheme

Theorem (4.2.9)

The path-following scheme terminates in at most N steps, where $N \leq O\left(\sqrt{\nu}\ln \frac{\nu\|c\|_{x_F^*}^*}{\epsilon}\right)$, with $x^T x_N - c^* \leq \epsilon$.

Proof.

(1) From Theorem 4.1.13, if $\lambda_F(x) < 1$ then $\|x - x_F^*\|_x \leq \omega_*'(\lambda_F(x))$, therefore $r_0 \equiv \|x_0 - x_F^*\|_{x_0} \leq \omega_*'(\lambda_F(x_0)) = \frac{\lambda_F(x_0)}{1-\lambda_F(x_0)} \leq \frac{\beta}{1-\beta} < 1$.

(2) Recall Theorem 4.1.6: *If $x \in \operatorname{dom} f$ for a self-concordant function f then for any $y \in W^0(x;1) = \{y \in \mathbb{R}^n : \|y - x\|_x < 1\}$ we have*

$$(1 - \|y - x\|_x)^2 f''(x) \preceq f''(y) \preceq \frac{1}{(1 - \|y - x\|_x)^2} f''(x).$$

Thus, for $f(x) = \frac{1}{2}[F''(x)]^{-1}, y = x_0, x = x_F^*$ we have

$$\|c\|_{x_0}^* = (c^T[F''(x_0)]^{-1}c)^{1/2} \leq \frac{1}{1 - \|x_0 - x_F^*\|_{x_F^*}}(c^T[F''(x_F^*)]^{-1}c)^{1/2} = \frac{1}{1 - r_0}\|c\|_{x_F^*}^*,$$

so

$$\frac{\gamma}{t_1} = \|c\|_{x_0}^* \leq \frac{1}{1 - r_0}\|c\|_{x_F^*}^* \leq \frac{1-\beta}{1-2\beta}\|c\|_{x_F^*}^*.$$

The general path-following scheme

(3) Since $t_k = t_{k-1} + \frac{\gamma}{\|c\|^*_{x_k}} \geq \left(1 + \frac{\gamma}{\beta+\sqrt{\nu}}\right) t_{k-1}$, we have

$$t_k \geq t_1 \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1} \geq \frac{\gamma(1-2\beta)}{(1-\beta)\|c\|^*_{x_F^*}} \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1}.$$

The stopping criterion is satisfied if

$$\frac{\gamma(1-2\beta)}{(1-\beta)\|c\|^*_{x_F^*}} \left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right)^{k-1} \geq \frac{1}{\epsilon}\left(\nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}\right), \ \textit{i.e.},$$

$$(k-1)\ln\left(1 + \frac{\gamma}{\beta + \sqrt{\nu}}\right) \geq \ln\left(\frac{(1-\beta)\|c\|^*_{x_F^*}}{\gamma(1-2\beta)}\frac{1}{\epsilon}\left(\nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}\right)\right),$$

Since $\ln(1+x) \geq \frac{x}{1+x}$ for any $x > -1$ the following guarantees that the stopping criterion is satisfied:

$$k-1 \geq \frac{\beta + \sqrt{\nu} + \gamma}{\gamma} \ln\left(\frac{(1-\beta)\|c\|^*_{x_F^*}}{\gamma(1-2\beta)}\frac{1}{\epsilon}\left(\nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}\right)\right) = O\left(\sqrt{\nu}\ln\frac{\nu\|c\|^*_{x_F^*}}{\epsilon}\right).$$

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The general path-following scheme

$$N \leq O\left(\sqrt{\nu}\ln\frac{\nu\|c\|^*_{x_F^*}}{\epsilon}\right).$$

Recall the inequality

$$\|x - x_F^*\|_{x_F^*} \leq \nu + 2\sqrt{\nu}$$

holding for any $x \in \mathrm{Dom}\ F$. Thus,

$$\|c^T(x - x_F^*)\|_{x_F^*} \leq \|c\|_{x_F^*}\|x - x_F^*\|_{x_F^*} \leq \|c\|_{x_F^*}(\nu + 2\sqrt{\nu}).$$

Hence, the value $\nu\|c\|^*_{x_F^*}$ estimates the variation of the linear function c^Tx over $\mathrm{Dom}\ F$, and the ratio $\frac{\epsilon}{\nu\|c\|^*_{x_F^*}}$ can be thought of as a relative accuracy of the solution.

Finding the analytic center

0. Set $t_0 = 0$. Choose an accuracy $\epsilon > 0$ and $x_0 \in \mathrm{dom}\ F$ such that

$$\|F'(x_0)\|^*_{x_0} \leq \beta.$$

1. k-th iteration ($k \geq 0$): Set

$$\begin{aligned} t_{k+1} &= t_k + \frac{\gamma}{\|c\|^*_{x_k}}, \\ x_{k+1} &= x_k - [F''(x_k)]^{-1}(t_{k+1}c + F'(x_k)). \end{aligned}$$

2. Stop if $\epsilon t_k \geq \nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}$.

We still need to figure out how to determine the *analytic center* of $\mathrm{Dom}\ F$ in step 0; i.e., we need to find an approximate solution $\bar{x} \in \mathrm{dom}\ F$ of the problem $\min\limits_{x \in \mathrm{dom}\ F} F(x)$ that satisfies the inequality $\|F'(\bar{x})\|^*_{\bar{x}} \leq \beta$. We will consider two methods: damped Newton and path following.

Finding the analytic center: damped Newton

0. Choose $y_0 \in \mathrm{dom}\ F$.
1. k-th iteration ($k \geq 0$). Set

$$y_{k+1} = y_k - \frac{[F''(y_k)]^{-1}F'(y_k)}{1 + \|F'(y_k)\|^*_{y_k}}.$$

2. Stop the process if $\|F'(y_k)\|^*_{y_k} \leq \beta$.

Finding the analytic center: damped Newton

Theorem (4.2.10)

The above damped Newton scheme terminates in at most $\frac{1}{\omega(\beta)}(F(y_0) - F(x_F^*))$ *iterations.*

Proof.

We have

$$F(y_{k+1}) \leq F(y_k) - \omega(\lambda_F(y_k)) \leq F(y_k) - \omega(\beta)$$

for any k before termination. Thus,

$$F(x_F^*) \leq F(y_{k+1}) \leq F(y_0) - k\omega(\beta)$$

and

$$k \leq \frac{1}{\omega(\beta)}(F(y_0) - F(x_F^*)).$$

□

Finding the analytic center: path following

For a fixed $y_0 \in \mathrm{dom}\, F$, define the auxiliary central path as

$$y^*(t) = \arg\min_{y \in \mathrm{dom}\, F}[-tF'(y_0)^T y + F(y)],$$

where $t \geq 0$. Then the equation

$$F'(y^*(t)) = tF'(y_0)$$

is satisfied. Note that for $t = 1$ we have

$$y^*(1) = \arg\min_{y \in \mathrm{dom}\, F}[-F'(y_0)^T y + F(y)] = y_0,$$

and

$$y^*(0) = \arg\min_{y \in \mathrm{dom}\, F}[F(y)] = x_F^*.$$

Thus, we can follow the trajectory $y^*(t)$ with decreasing t.

Finding the analytic center: path following

We obtain the following auxiliary path-following scheme:

0. Choose $y_0 \in \mathrm{Dom}\, F$. Set $t_0 = 1$.
1. k-th iteration ($k \geq 0$): Set

$$\begin{aligned} t_{k+1} &= t_k - \frac{\gamma}{\|F'(y_0)\|_{y_k}^*}, \\ y_{k+1} &= y_k - [F''(y_k)]^{-1}(t_{k+1}F'(y_0) + F'(y_k)). \end{aligned}$$

2. Stop if $\|F'(y_k)\|_{y_k}^* \leq \frac{\sqrt{\beta}}{1+\sqrt{\beta}}$.
   Set $\bar{x} = y_k - [F''(y_k)]^{-1}F'(y_k)$.

Finding the analytic center: path following

Recall the following theorem:

Theorem (4.1.14)

Let $x \in \mathrm{dom}\, f$ *and* $\lambda_f(x) < 1$. *Then the point*

$$x_+ = x - [f''(x)]^{-1}f'(x)$$

belongs to $\mathrm{dom}\, f$ *and* $\lambda_f(x_+) \leq \left(\frac{\lambda_f(x)}{1-\lambda_f(x)}\right)^2$.

Since the stopping criterion of the auxiliary path-following scheme is given by

$$\lambda_k = \|F'(y_k)\|_{y_k}^* \leq \frac{\sqrt{\beta}}{1+\sqrt{\beta}},$$

this guarantees that

$$\|F'(\bar{x})\|_{\bar{x}}^* \leq \left(\frac{\lambda_k}{1-\lambda_k}\right)^2 = \beta.$$

Finding the analytic center: path following

Theorem (4.2.11)

The auxiliary path-following scheme terminates in at most

$$\frac{1}{\gamma}(\beta+\sqrt{\nu})\ln\left[\frac{1}{\gamma}(\nu+2\sqrt{\nu})\|F'(x_0)\|^*_{x_F^*}\right]$$

iterations.

Proof.

(1) Recall that $\beta = 1/9,\ \gamma = \sqrt{\beta}/(1+\sqrt{\beta}) - \beta = 5/36,\ t_0 = 1$.

(2) We have

$$t_{k+1} \le \left(1-\frac{\gamma}{\beta+\sqrt{\nu}}\right)t_k \le \left(1-\frac{\gamma}{\beta+\sqrt{\nu}}\right)^{k+1} t_0 \le \exp\left(-\frac{\gamma(k+1)}{\beta+\sqrt{\nu}}\right).$$

(3) For any $t \ge 0$ we have

$$\|F'(y^*(t))\|^*_{y^*(t)} = \|tF'(y(0))\|^*_{y^*(t)} \le (\nu+2\sqrt{\nu})\|F'(y_0)\|^*_{x_F^*}t, \text{ thus}$$

$$\begin{aligned} \|F'(y_k)\|^*_{y_k} &= \|(t_kF'(x_0)+F'(y_k)) - t_kF'(x_0)\|^*_{y_k} \\ &\le \beta + t_k\|F'(x_0)\|^*_{y_k} \le \beta + t_k(\nu+2\sqrt{\nu})\|F'(x_0)\|^*_{x_F^*}. \end{aligned}$$

Finding the analytic center: path following

Thus, the process will terminate if the following inequality holds:

$$t_k(\nu+2\sqrt{\nu})\|F'(x_0)\|^*_{x_F^*} \le \frac{\sqrt{\beta}}{1+\sqrt{\beta}} - \beta = \gamma,$$

i.e., in at most

$$\frac{1}{\gamma}(\beta+\sqrt{\nu})\ln\left[\frac{1}{\gamma}(\nu+2\sqrt{\nu})\|F'(x_0)\|^*_{x_F^*}\right]$$

iterations ($O(\sqrt{\nu})[\ln\nu + \ln\|F'(x_0)\|^*_{x_F^*}]$). $\square$

Complexity of the general path-following scheme

0. Set $t_0 = 0$. Choose an accuracy $\epsilon > 0$ and $x_0 \in \operatorname{dom} F$ such that
$$\|F'(x_0)\|^*_{x_0} \le \beta.$$
1. k-th iteration ($k \ge 0$): Set
$$\begin{aligned} t_{k+1} &= t_k + \tfrac{\gamma}{\|c\|^*_{x_k}}, \\ x_{k+1} &= x_k - [F''(x_k)]^{-1}(t_{k+1}c + F'(x_k)). \end{aligned}$$
2. Stop if $\epsilon t_k \ge \nu + \frac{(\beta+\sqrt{\nu})\beta}{1-\beta}$.

Complexity of the general path-following scheme

- Step 0 terminates in $O(\sqrt{\nu})[\ln\nu + \ln\|F'(x_0)\|^*_{x_F^*}]$ steps (using the auxiliary path-following scheme).
- Step 1 terminates in $O\left(\sqrt{\nu}\ln\frac{\nu\|c\|^*_{x_F^*}}{\epsilon}\right)$ steps.

The total run time is

$$\boxed{O\left(\sqrt{\nu}\left[\ln\nu + \ln\|F'(x_0)\|^*_{x_F^*} + \ln\|c\|^*_{x_F^*} + \ln\frac{1}{\epsilon}\right]\right).}$$