ISEN 629: Engineering Optimization

Lecture 23

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Problems with functional constraints

We consider the following problem:

$$\begin{array}{rl} & \min\ f_0(x), \\ s.t. & f_j(x) \leq 0, j = 1, \ldots, m, \\ & x \in Q, \end{array} \tag{1}$$

where Q is a simple compact convex set with nonempty interior and all functions $f_j, j = 0, \ldots, m$, are convex.
We also assume that Slater condition is satisfied: There exists $\bar{x} \in \mathrm{int}\ Q$ such that $f_j(\bar{x}) < 0$ for all $j = 1, \ldots, m$.

Problems with functional constraints

Assume that we know an upper bound $\bar{\tau}$ such that $f_0(x) < \bar{\tau}$ for all $x \in Q$. Introducing two additional variables τ and κ, our problem can be equivalently written in the standard form:

$$\begin{array}{rl} & \min\ \tau, \\ s.t. & f_0(x) \leq \tau, \\ & f_j(x) \leq \kappa, j = 1, \ldots, m, \\ & x \in Q, \tau \leq \bar{\tau}, \kappa \leq 0. \end{array} \tag{2}$$

In order to apply the interior-point methods to this problem, we need to be able to construct a self-concordant barrier for the feasible set, consisting of

- A self-concordant barrier $F_Q(x)$ for the set Q;
- A self-concordant barrier $F_0(x, \tau)$ for $\mathrm{epi}\ f_0$;
- Self-concordant barriers $F_j(x, \kappa)$ for $\mathrm{epi}\ f_j, j = 1, \ldots, m$.

Problems with functional constraints

Then the resulting barrier is given by

$$\hat{F}(x, \tau, \kappa) = F_Q(x) + F_0(x, \tau) + \sum_{j=1}^{m} F_j(x, \kappa) - \ln(\bar{\tau} - \tau) - \ln(-\kappa),$$

with the parameter

$$\hat{\nu} = \nu_Q + \nu_0 + \sum_{j=1}^{m} \nu_j + 2,$$

where $\nu_Q, \nu_j, j = 1, \ldots, m$ are the parameters of the corresponding barriers.

We need to find a way to determine a starting point from $\mathrm{dom}\ \hat{F}$.

Problems with functional constraints

We change the notation to obtain the problem

$$\begin{array}{rl} & \min \ c^T z, \\ s.t. & z \in S, \\ & d^T z \le 0, \end{array} \qquad (3)$$

where $z = (x, \tau, \kappa), c^T z \equiv \tau, d^T z \equiv \kappa$ and

$$S = \{x \in Q : f_0(x) \le \tau, f_j(x) \le \kappa, j = 1, \dots, m, \tau \le \bar{\tau}\}$$

(i.e., the feasible set of our problem without the constraint $\kappa \le 0$).

We know a self-concordant barrier $F(z)$ for the set S (assuming that we know barriers for all constraints) and we can easily find a point $z_0 \in \mathrm{int}\ S$ by selecting τ_0 and κ_0 large enough to guarantee

$$f_0(x_0) < \tau_0 < \bar{\tau}, \ f_j(x_0) < \kappa_0, j = 1, \dots, m.$$

In addition, the set

$$S(\alpha) = \{z \in S : d^T z \le \alpha\}$$

is bounded and has a nonempty interior if α is large enough.

Problems with functional constraints

To solve our problem, we will use a three-stage process:

1. Find an approximate analytic center of the set $S(\alpha)$.
2. Find an approximate analytic center $\bar{z}$ of the set $\bar{S} = \{z \in S(\alpha) : d^T z \le 0\}$.
3. Apply the main path-following scheme starting with $\bar{z}$ to solve the problem

   $$\min\{c^T z : z \in \bar{S}\},$$

   which is equivalent to problem (3):

   $$\begin{array}{rl} & \min \ c^T z, \\ s.t. & z \in S, \\ & d^T z \le 0, \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} & \min \ c^T z, \\ s.t. & z \in S(\alpha), \\ & d^T z \le 0, \end{array}$$

   since $S(\alpha) = \{z \in S : d^T z \le \alpha\}$.

Problems with functional constraints

1. Choose a starting point $z_0 \in \mathrm{int}\ S$, $\Delta > 0$ and set $\alpha = d^T z_0 + \Delta$. If $\alpha \le 0$, then we have a feasible point and we can use the two-stage path-following method discussed before. Otherwise, we find an approximate analytic center of the set $S(\alpha)$ generated by the barrier

   $$\tilde{F}(z) = F(z) - \ln(\alpha - d^T z).$$

   This can be done using the auxiliary path following scheme. The approximate analytic center $\tilde{z}$ will satisfy the condition

$$\lambda_{\tilde{F}}(\tilde{z}) \equiv \left[\left(F'(\tilde{z}) + \frac{d}{\alpha - d^T \tilde{z}} \right)^T \tilde{F}''(\tilde{z})^{-1} \left(F'(\tilde{z}) + \frac{d}{\alpha - d^T \tilde{z}} \right) \right]^{1/2} \le \beta.$$

Problems with functional constraints

2. To find an approximate analytic center $\bar{z}$ of the set $\bar{S} = \{z \in S(\alpha) : d^T z \le 0\} \equiv \{z \in S : d^T z \le 0\}$ generated by the barrier

   $$\bar{F}(z) = \tilde{F}(z) - \ln(-d^T z),$$

   we solve the minimization problem $\min\{d^T z : z \in S(\alpha)\}$ by following the central path $z(t)$ defined by the equation

   $$td + \tilde{F}'(z(t)) = 0, t \ge 0.$$

   Due to the Slater condition, the optimal value will be negative. Since the analytic center z^* satisfies the equation $\tilde{F}'(z^*) - \frac{d}{d^T z^*} = 0$, it is a point of the central path $z(t)$ (with $t^* = -1/(d^T z^*) > 0$). As a result, we find $\bar{z}$ such that

   $$\lambda_{\bar{F}}(\bar{z}) \equiv \left[\left(\tilde{F}'(\bar{z}) - \frac{d}{d^T \bar{z}} \right)^T \bar{F}''(\bar{z})^{-1} \left(\tilde{F}'(\bar{z}) - \frac{d}{d^T \bar{z}} \right) \right]^{1/2} \le \beta.$$

Problems with functional constraints

3. Finally, starting with $\bar{z}$, we apply the main path-following scheme to solve the problem

$$\min\{c^T z : z \in \bar{S}\},$$

which is equivalent to the original problem (3).

To summarize, we have developed efficient interior point methods for problems, for which we know self-concordant barriers for Q and epigraphs of the objective and the functional constraints.

Later we will describe classes of problems for which such barriers can be provided.