ISEN 629: Engineering Optimization

Lecture 24

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Bounds on parameters of self-concordant barriers

Lemma (4.3.1)

Let $f(t)$ be a ν-self-concordant barrier for the interval $(\alpha, \beta) \subset \mathbb{R}$, $\alpha < \beta < \infty$. Then $\nu \geq \kappa \equiv \sup\limits_{t \in (\alpha,\beta)} \frac{f'(t)^2}{f''(t)} \geq 1$.

Proof.

Note that $\mu \geq \kappa$ by the definition of self-concordant barrier. Assume that $\kappa < 1$. Since $f(t)$ is a barrier for (α, β), there exists $\bar{\alpha} \in (\alpha, \beta)$ such that $f'(t) > 0$ for all $t \in [\bar{\alpha}, \beta)$. Then for $\phi(t) = \frac{f'(t)^2}{f''(t)}$, $t \in [\bar{\alpha}, \beta)$ we have

$$\begin{aligned} \phi'(t) &= 2f'(t) - \left(\frac{f'(t)}{f''(t)}\right)^2 f'''(t) \\ &= f'(t)\left(2 - \frac{f'(t)}{\sqrt{f''(t)}} \frac{f'''(t)}{[f''(t)]^{3/2}}\right) \geq 2(1 - \sqrt{\kappa}) f'(t). \end{aligned}$$

Hence, by mean value theorem, for all $t \in [\bar{\alpha}, \beta)$ we have

$$\phi(t) \geq \phi(\bar{\alpha}) + 2(1 - \sqrt{\kappa})(f(t) - f(\bar{\alpha})).$$

Since $f(t)$ is a barrier, this contradicts to the assumption that $\phi(t)$ is bounded from above. □

Bounds on parameters of self-concordant barriers

Corollary

Let $F(x)$ be a ν-self-concordant barrier for $Q \subset \mathbb{R}^n$. Then $\nu \geq 1$.

Proof.

Let $x \in \operatorname{int} Q$. Q is a strict subset of $\mathbb{R}^n$, so there exists a nonzero direction $u \in \mathbb{R}^n$ such that the line $\{y = x + tu, t \in \mathbb{R}\}$ intersects the boundary of Q. Therefore, applying the above lemma for the function $f(t) = F(x + tu)$ we obtain the result. □

Bounds on parameters of self-concordant barriers

Definition

A direction d is called a recession direction of the set Q at point x if

$$x + \alpha d \in Q \ \ \forall \alpha \geq 0.$$

Let Q be a closed convex set with nonempty interior, and let $\bar{x} \in \operatorname{int} Q$. Assume that there exists a set of recession directions $\{p_1, \dots, p_k\}$ of Q

Theorem (4.3.1)

Let positive coefficients $\{\beta_i\}_{i=1}^k$ satisfy the condition $\bar{x} - \beta_i p_i \notin \operatorname{int} Q$, $i = 1, \dots, k$. If for some positive $\alpha_1, \dots, \alpha_k$ we have $\bar{y} = \bar{x} - \sum\limits_{i=1}^k \alpha_i p_i \in Q$, then the parameter ν of any self-concordant barrier for Q satisfies the inequality $\nu \geq \sum\limits_{i=1}^k \frac{\alpha_i}{\beta_i}$.

Existence theorem for self-concordant barriers

Consider a closed convex set Q with nonempty interior, and assume that Q contains no straight line.
For $\bar{x} \in \mathrm{int}\, Q$, the set

$$P(\bar{x}) = \{s \in \mathbb{R}^n : s^T(x - \bar{x}) \leq 1, \ \forall x \in Q\}$$

is called a polar set of Q with respect to $\bar{x}$. It can be shown that for any $x \in \mathrm{int}\, Q$, the set $P(x)$ is a compact convex set with nonempty interior.

Theorem (4.3.2)

There exist constants c_1 and c_2 such that the function

$$U(x) = c_1 \cdot \ln\left(\mathrm{vol}_n P(x)\right)$$

is a $(c_2 n)$-self-concordant barrier for Q, which is called the universal barrier for the set Q.

Analytical complexity bound

Recall that the problem

$$\min_{x \in Q} c^T x,$$

where Q is a closed convex set with a nonempty interior, for which we know a ν-self-concordant barrier $F(x)$, can be solved using

$$O\left(\sqrt{\nu} \ln \frac{\nu}{\epsilon}\right)$$

iterations of a path-following scheme.

Linear optimization

Consider the problem

$$\begin{aligned} & \min c^T x, \\ s.t. \quad & Ax = b, \\ & x \geq 0 \ (\Leftrightarrow x \in \mathbb{R}^n_+), \end{aligned}$$

where A is an $m \times n$-matrix, $m < n$.
We can use the standard logarithmic barrier for $\mathbb{R}^n_+$:

$$F(x) = -\sum_{i=1}^n \ln x_i, \ \nu = n.$$

Since the restriction of the barrier $F(x)$ onto affine subspace $\{x : Ax = b\}$ is an n-self-concordant barrier, this problem can be solved in $O\left(\sqrt{n} \ln \frac{n}{\epsilon}\right)$ iterations of a path-following scheme.

Linear optimization

Next we show that the standard logarithmic barrier is optimal for $\mathbb{R}^n_+$.

Lemma (4.3.2)

Parameter ν of any self-concordant barrier for the positive orthant $\mathbb{R}^n_+$ cannot be less than n.

Proof.

Consider

$$\begin{aligned} \bar{x} &= e \equiv (1, \ldots, 1)^T \in \mathrm{int}\, \mathbb{R}^n_+, \\ p_i &= e_i, \ i = 1, \ldots, n, \end{aligned}$$

where e_i is the i-th orth in $\mathbb{R}^n$. Using Theorem 4.3.1 with $\alpha_i = \beta_i = 1, i = 1, \ldots, n$ we obtain

$$\nu \geq \sum_{i=1}^n \frac{\alpha_i}{\beta_i} = n.$$

□

Quadratic optimization

Consider a quadratically constrained quadratic problem

$$\begin{array}{rl} & \min q_0(x) = \frac{1}{2}x^T A_0 x + b^T x + c, \\ s.t. & q_i(x) = \frac{1}{2}x^T A_i x + b_i^T x + c_i \le d_i, \; i = 1, \dots, m, \end{array}$$

where A_i's are positive semidefinite $n \times n$-matrices. We write this problem in the standard form:

$$\begin{array}{rl} & \min \tau, \\ s.t. & q_0(x) \le \tau, \\ & q_i(x) \le d_i, i = 1, \dots, m, \\ & x \in \mathbb{R}^n, \tau \in \mathbb{R}. \end{array}$$

The following function is a self-concordant barrier for this problem:

$$F(x,\tau) = -\ln(\tau - q_0(x)) - \sum_{i=1}^{m} \ln(d_i - q_i(x)), \; \nu = m + 1.$$

Thus, this problem can be solved in $O(\sqrt{m+1} \ln \frac{m}{\epsilon})$ iterations of a path-following scheme.

Second-order cones

The convex set $K_2 = \{(x,t) \in \mathbb{R}^{n+1} : t \ge \|x\|\}$ is called a second-order cone.

Lemma (4.3.3)

The function

$$F(x,t) = -\ln(t^2 - \|x\|^2)$$

is a 2-self-concordant barrier for the second-order cone K_2.

This choice of $F(x,t)$ appears to be optimal:

Lemma (4.3.4)

Any ν*-self-concordant barrier for* K_2 *has the parameter* $\nu \ge 2$.

Semidefinite optimization

In semidefinite optimization the decision variables are matrices.

Denote by $S^{n\times n}$ the linear space of symmetric $n \times n$-matrices.

Consider the Frobenius inner product for $S^{n\times n}$: for any $X, Y \in S^{n\times n}$

$$\langle X, Y\rangle_F = \sum_{i=1}^{n}\sum_{j=1}^{n} X^{(i,j)} Y^{(i,j)} = \mathrm{trace}(XY),$$

and the Frobenius norm of X defined by

$$\|X\|_F = \langle X, X\rangle_F^{1/2}.$$

In semidefinite optimization we require that the matrix variable is positive semidefinite, i.e. $X \in \mathcal{P}_n$, where $\mathcal{P}_n \subset S^{n\times n}$ is the cone of positive semidefinite $n \times n$-matrices.

Semidefinite optimization

A semidefinite program is formulated as follows:

$$\begin{array}{rl} & \min \langle C, X\rangle_F, \\ s.t. & \langle A_i, X\rangle_F = b_i, \; i = 1, \dots, m, \\ & X \in \mathcal{P}_n, \end{array} \qquad (1)$$

where C and A_i are some matrices from $S^{n\times n}$.

To apply a path-following approach, we need a barrier for $\mathcal{P}_n$.

Let $X \in \mathrm{int}\, \mathcal{P}_n$ (i.e., X is positive definite). Denote by

$$F(X) = -\ln\det(X) \equiv -\ln \prod_{i=1}^{n} \lambda_i(X),$$

where $\{\lambda_i(X)\}_{i=1}^n$ is the set of eigenvalues of X.

Semidefinite optimization

Lemma (4.3.5)

Function $F(X)$ is convex and $F'(X) = -X^{-1}$. For any direction $\Delta \in S^{n\times n}$ we have

$$\begin{aligned} \langle F''(X)\Delta, \Delta\rangle_F &= \|X^{-1/2}\Delta X^{-1/2}\|_F^2 = \langle X^{-1}\Delta X^{-1}, \Delta\rangle_F \\ &= \mathrm{trace}\left([X^{-1/2}\Delta X^{-1/2}]^2\right), \\ D^3F(x)[\Delta,\Delta,\Delta] &= -2\langle I_n, [X^{-1/2}\Delta X^{-1/2}]^3\rangle_F \\ &= -2\mathrm{trace}\left([X^{-1/2}\Delta X^{-1/2}]^3\right). \end{aligned}$$

Semidefinite optimization

Theorem (4.3.3)

Function $F(X)$ is an n-self-concordant barrier for $\mathcal{P}_n$.

Lemma (4.3.6)

For any ν-self-concordant barrier of the cone $\mathcal{P}_n$ we have $\nu \geq n$.

Semidefinite optimization

Since our problem is

$$\begin{aligned} & \min \langle C, X\rangle_F, \\ s.t. \quad & \langle A_i, X\rangle_F = b_i, \ \ i = 1,\dots,m, \\ & X \in \mathcal{P}_n, \end{aligned}$$

we need to use a restriction of $F(X)$ onto the set

$$\mathcal{L} = \{X : \langle A_i, X\rangle_F = b_i, i = 1,\dots,m\},$$

which is still an n-self-concordant barrier.

Thus, we can obtain an ϵ-optimal solution to our problem in $O(\sqrt{n}\ln\frac{n}{\epsilon})$ iterations of a path-following scheme.

Note that we have $O(n^2)$ variables.

Let us estimate the per-iteration complexity.

Semidefinite optimization

The Newton step consists in minimizing a quadratic approximation of our problem at some $X \succ 0$ belonging to $\mathcal{L}$, which in our case is

$$\min_{\Delta}\{\langle U, \Delta\rangle_F + \frac{1}{2}\langle X^{-1}\Delta X^{-1}, \Delta\rangle_F : \langle A_i, \Delta\rangle_F = 0, i = 1,\dots,m\},$$

where $U = C + F'(X) = C - X^{-1}$. FONC yields

$$\begin{aligned} U + X^{-1}\Delta X^{-1} &= \sum_{j=1}^m \lambda^{(j)} A_j, \\ \langle A_i, \Delta\rangle_F &= 0, i = 1,\dots,m. \end{aligned}$$

Expressing Δ from the first equation,

$$\Delta = X\left[-U + \sum_{j=1}^m \lambda^{(j)} A_j\right] X,$$

and substituting into the second, we obtain a linear system

$$\sum_{j=1}^m \lambda^{(j)}\langle A_i, XA_jX\rangle_F = \langle A_i, XUX\rangle_F, i = 1,\dots,m.$$

Semidefinite optimization

$$\sum_{j=1}^{m} \lambda^{(j)} \langle A_i, XA_jX \rangle_F = \langle A_i, XUX \rangle_F, i = 1, \ldots, m.$$

Denoting by

$$S^{(i,j)} = \langle A_i, XA_jX \rangle_F,\ d^{(j)} = \langle U, XA_iX \rangle_F,\ i, j = 1, \ldots, m,$$

we obtain the system $S\lambda = d$.

- Computing matrices $XA_jA, j = 1, \ldots, m$ takes $O(mn^3)$ operations.
- Computing S and d takes $O(m^2n^2)$ operations.
- Computing $\lambda = S^{-1}d$ takes $O(m^3)$ operations.
- Computing Δ takes $O(mn^2)$ operations.

Since $m = O(n^2)$, the total cost is $O(mn^2(m + n))$ operations.

Semidefinite optimization

The barrier $-\ln\det(\cdot)$ can be used for some functions of eigenvalues.

For example, consider a matrix $\mathcal{A}(x) \in S^{n\times n}$, which depends linearly on x. Then the convex region

$$\{(x, t) : \max_{1\le i\le n} \lambda_i(\mathcal{A}(x)) \le t\}$$

can be described by a self-concordant barrier

$$F(x, t) = -\ln\det(tI_n - \mathcal{A}(x))$$

with $\nu = n$.