ISEN 629: Engineering Optimization

Lecture 25

Sergiy Butenko

Industrial and Systems Engineering
Texas A&M University

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Preliminaries: n-dimensional volumes

Lemma
If a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix A and S is any measurable subset of $\mathbb{R}^n$, then the m-dimensional volume of $f(S)$ is given by $\sqrt{\det(A^TA)} \times \mathrm{vol}_n(S)$.

In particular, if a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by matrix A and S is any measurable subset of $\mathbb{R}^n$, then the volume of $f(S)$ is given by $|\det A| \times \mathrm{vol}_n(S)$.

Circumscribed ellipsoid

Problem
Given a set of points $a_1, \ldots, a_m \in \mathbb{R}^n$, find an ellipsoid W of minimum volume containing all these points.

Solution
We can represent any ellipsoid as $W = \{x \in \mathbb{R}^n : \|Hx - v\| \le 1\}$, where $H \in \mathrm{int}\mathcal{P}_n$ and $v \in \mathbb{R}^n$. Then

$$W = \{x \in \mathbb{R}^n : \|Hx - v\| \le 1\} \equiv \{x \in \mathbb{R}^n : x = H^{-1}(v + u), \|u\| \le 1\}.$$

Note that

$$\mathrm{vol}_n(W) = \mathrm{vol}_n(B(0,1)) \cdot \det H^{-1} = \frac{\mathrm{vol}_n(B(0,1))}{\det H},$$

so minimizing the volume of W is equivalent to minimizing

$$-\ln\det H.$$

Circumscribed ellipsoid

Thus, we can formulate our problem as follows:

$$\begin{array}{ll} & \min\limits_{H,v,\tau} \tau, \\ \text{s.t.} & -\ln\det H \le \tau, \\ & \|Ha_i - v\| \le 1, i = 1, \ldots, m, \\ & H \in \mathcal{P}_n, v \in \mathbb{R}^n, \tau \in \mathbb{R}. \end{array}$$

Lemma (4.3.7)
The function

$$-\ln\det H - \ln(\tau + \ln\det H)$$

is an $(n+1)$-self-concordant barrier for the set

$$\{(H, \tau) \in \mathcal{S}^{n\times n} \times \mathbb{R} : \tau \ge -\ln\det H, H \in \mathcal{P}_n\}.$$

Thus, we can use the following barrier to solve our problem:

$$F(H, v, \tau) = -\ln\det H - \ln(\tau + \ln\det H) - \sum_{i=1}^{m} \ln(1 - \|Ha_i - v\|^2)$$

with $\nu = m + n + 1$.

Inscribed ellipsoid with fixed center

Problem

Given $Q = \{x \in \mathbb{R}^n : a_i^T x \le b_i, i = 1, \dots, m\}$ and $v \in \mathrm{int} Q$. Find a maximum volume ellipsoid $W \subset Q$ centered at v.

Solution

Lemma (4.3.8)

Let $a^T v < b$. Inequality $a^T x \le b$ is valid for any $x \in W$ if and only if $a^T H a \le (b - a^T v)^2$.

The ellipsoid sought is in the form

$$W = \{x \in \mathbb{R}^n : (x - v)^T H^{-1} (x - v) \le 1\}.$$

Note that $\mathrm{vol}_n(W) = \mathrm{vol}_n(B(0,1))(\det H)^{1/2}$, so the problem of maximizing $\mathrm{vol}_n(W)$ is equivalent to the problem of minimizing $-\ln \det H$.

Inscribed ellipsoid with fixed center

Thus, we obtain:

$$\begin{array}{ll} & \min_{H,\tau} \tau, \\ \text{s.t.} & -\ln \det H \le \tau, \\ & a_i^T H a_i \le (b_i - a_i^T v)^2, i = 1, \dots, m, \\ & H \in \mathcal{P}_n, \tau \in \mathbb{R}. \end{array}$$

We can use the following self-concordant barrier:

$$\begin{aligned} F(H, \tau) &= -\ln \det H - \ln(\tau + \ln \det H) \\ &\quad - \sum_{i=1}^m \ln[(b_i - a_i^T v)^2 - a_i^T H a_i] \end{aligned}$$

with $\nu = m + n + 1$. Thus, we can solve the problem in $O(\sqrt{m+n+1} \ln \frac{m+n}{\epsilon})$ iterations of a path-following scheme.

Inscribed ellipsoid with free center

Problem

Given $Q = \{x \in \mathbb{R}^n : a_i^T x \le b_i, i = 1, \dots, m\}$ with $\mathrm{int} Q \ne \emptyset$, find a maximum volume ellipsoid $W \subset Q$.

Solution

Let $G \in \mathrm{int} \mathcal{P}_n, v \in \mathrm{int} Q$. We can represent W as follows:

$$\begin{aligned} W &= \{x \in \mathbb{R}^n : \|G^{-1}(x - v)\| \le 1\} \\ &\equiv \{x \in \mathbb{R}^n : (x - v)^T G^{-2} (x - v) \le 1\}. \end{aligned}$$

Note that the inequality $a^T x \le b$ is valid for any $x \in W$ if and only if

$$\|Ga\|^2 \equiv a^T G^2 a \le (b - a^T v)^2,$$

yielding a convex region for (G, v): $\|Ga\| \le b - a^T v$. Note that

$$\mathrm{vol}_n(W) = \mathrm{vol}_n(B(0,1)) \det G.$$

Inscribed ellipsoid with free center

Thus, we obtain:

$$\begin{array}{ll} & \min_{G,v,\tau} \tau, \\ \text{s.t.} & -\ln \det G \le \tau, \\ & \|Ga_i\| \le b_i - a_i^T v, i = 1, \dots, m, \\ & a_i^T v \le b_i, i = 1, \dots, m \\ & G \in \mathcal{P}_n, v \in \mathbb{R}^n, \tau \in \mathbb{R}. \end{array}$$

We can use the following self-concordant barrier:

$$\begin{aligned} F(G, v, \tau) &= -\ln \det G - \ln(\tau + \ln \det G) \\ &\quad - \sum_{i=1}^m \ln[(b_i - a_i^T v)^2 - \|Ga_i\|^2] \end{aligned}$$

with $\nu = 2m + n + 1$. Thus, we can solve the problem in $O(\sqrt{2m+n+1} \ln \frac{m+n}{\epsilon})$ iterations of a path-following scheme.

Separable optimization

Consider a general separable problem:

$$\min_{x\in\mathbb{R}^n} q_0(x) = \sum_{j=1}^{m_0} \alpha_{0j} f_{0j}(a_{0j}^T x + b_{0j}),$$
$$\text{s.t.} \quad q_i(x) = \sum_{j=1}^{m_i} \alpha_{ij} f_{ij}(a_{ij}^T x + b_{ij}) \le \beta_i, \; i = 1, \ldots, m,$$

where α_{ij} are some positive coefficients, $a_{ij} \in \mathbb{R}^n$ and $f_{ij}(t)$ are univariate convex functions. Put it in the standard form:

$$\begin{aligned} & \min_{x,t,\tau} \tau_0, \\ \text{s.t.} \quad & f_{ij}(a_{ij}^T x + b_{ij}) \le t_{ij}, i = 0, \ldots, m, j = 1, \ldots, m_i, \\ & \sum_{j=1}^{m_i} \alpha_{ij} t_{ij} \le \tau_i, i = 0, \ldots, m, \\ & \tau_i \le \beta_i, i = 1, \ldots, m, \\ & x \in \mathbb{R}^n, \tau \in \mathbb{R}^{m+1}, t \in \mathbb{R}^M, \text{where } M = \sum_{i=0}^{m} m_i. \end{aligned}$$

We need barriers for epigraphs of f_{ij}...

Barriers for some univariate functions

Logarithm and exponent

$F_1(x,t) = -\ln x - \ln(\ln x + t)$ is a 2-self-concordant barrier for the set

$$Q_1 = \{(x,t) \in \mathbb{R}^2 : x > 0, t \ge -\ln x\},$$

and $F_2(x) = -\ln t - \ln(\ln t - x)$ is a 2-self-concordant barrier for the set

$$Q_2 = \{(x,t) \in \mathbb{R}^2 : t \ge e^x\}.$$

Entropy function

$F_3(x,t) = -\ln x - \ln(t - x\ln x)$ is a 2-self-concordant barrier for the set

$$Q_3 = \{(x,t) \in \mathbb{R}^2 : x \ge 0, t \ge x\ln x\}.$$

Barriers for some univariate functions

Increasing power functions

$F_4(x,t) = -2\ln t - \ln(t^{2/p} - x^2)$ is a 4-self-concordant barrier for the set

$$Q_4 = \{(x,t) \in \mathbb{R}^2 : t \ge |x|^p\}, p \ge 1,$$

and $F_5(x) = -\ln x - \ln(t^p - x)$ is a 2-self-concordant barrier for the set

$$Q_5 = \{(x,t) \in \mathbb{R}^2 : x \ge 0, t^p \ge x\}, 0 < p \le 1.$$

Decreasing power functions

$F_6(x,t) = -\ln t - \ln(x - t^{-1/p})$ is a 2-self-concordant barrier for the set

$$Q_6 = \left\{(x,t) \in \mathbb{R}^2 : x > 0, t \ge \frac{1}{x^p}\right\}, p \ge 1,$$

and $F_7(x) = -\ln x - \ln(t - x^{-p})$ is a 2-self-concordant barrier for the set

$$Q_7 = \left\{(x,t) \in \mathbb{R}^2 : x > 0, t \ge \frac{1}{x^p}\right\}, 0 < p \le 1.$$

Geometric optimization

Consider the problem

$$\begin{aligned} & \min_{x\in\mathbb{R}^n} q_0(x) = \sum_{j=1}^{m_0} \alpha_{0j} \prod_{r=1}^{n} (x^{(r)})^{\sigma_{0j}^{(r)}}, \\ \text{s.t.} \quad & q_i(x) = \sum_{j=1}^{m_i} \alpha_{ij} \prod_{r=1}^{n} (x^{(r)})^{\sigma_{ij}^{(r)}} \le 1, i = 1, \ldots, m, \\ & x^{(j)} > 0, j = 1, \ldots, n, \end{aligned}$$

where α_{ij} are some positive coefficients. This problem is not convex. Denoting by $a_{ij} = (\sigma_{ij}^{(1)}, \ldots, \sigma_{ij}^{(n)}) \in \mathbb{R}^n$ and changing the variables:

$$x^{(i)} = e^{y^{(i)}},$$

we obtain an equivalent convex separable problem

$$\begin{aligned} & \min_{y\in\mathbb{R}^n} \sum_{j=1}^{m_0} \alpha_{0j} \exp\{a_{0j}^T y\}, \\ \text{s.t.} \quad & \sum_{j=1}^{m_i} \alpha_{ij} \exp\{a_{ij}^T y\} \le 1, i = 1, \ldots, m. \end{aligned}$$

Geometric optimization

Denote by $M = \sum_{i=0}^{m} m_i$. Then our problem can be solved in $O\left(M^{1/2} \cdot \ln \frac{M}{\epsilon}\right)$ iterations of a path-following scheme.

Approximation in l_p norms

The problem

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} |a_i^T x - b^{(i)}|^p, \\ \text{s.t.} \quad & \alpha \leq x \leq \beta, \end{aligned}$$

where $p \geq 1$, is equivalent to the following problem

$$\begin{aligned} & \min_{x,\tau} \tau^{(0)}, \\ \text{s.t.} \quad & |a_i^T x - b^{(i)}|^p \leq \tau^{(i)}, i = 1, \ldots, m, \\ & \sum_{i=1}^{m} \tau^{(i)} \leq \tau^{(0)}, \\ & \alpha \leq x \leq \beta, \\ & x \in \mathbb{R}^n, \tau \in \mathbb{R}^{m+1}. \end{aligned}$$

Approximation in l_p norms

We consider the problem

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} |a_i^T x - b^{(i)}|^p, \\ \text{s.t.} \quad & \alpha \leq x \leq \beta, \end{aligned}$$

where $p \geq 1$.
The total complexity of two alternative approaches to solving this problem is given by:

- Ellipsoid method: $O\left(n^3(m+n) \ln \frac{1}{\epsilon}\right)$ operations;
- Path-following method: $O\left(n^2(m+n)^{3/2} \ln \frac{m+n}{\epsilon}\right)$ operations.

Thus, interior-point methods are more efficient if $m \leq O(n^2)$.