Graph Domination, Coloring and Cliques in Telecommunications

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Abstract

This paper aims to provide a detailed survey of existing graph models and algorithms for important problems that arise in different areas of wireless telecommunication. In particular, applications of graph optimization problems such as minimum dominating set, minimum vertex coloring and maximum clique in multihop wireless networks are discussed. Different forms of graph domination have been used extensively to model clustering in wireless ad hoc networks. Graph coloring problems and their variants have been used to model channel assignment and scheduling type problems in wireless networks. Cliques are used to derive bounds on chromatic number, and are used in models of traffic flow, resource allocation, interference, etc. In this paper we survey the solution methods proposed in the literature for these problems and some recent theoretical results that are relevant to this area of research in wireless networks.

Keywords: Dominating sets, independent sets, cliques, coloring, wireless networks.

1 Introduction

The telecommunication industry has always been a host for a wide variety of optimization problems. In recent years, extensive research has led to the rapid development of numerous mobile computing devices that are diverse and technologically intensive. This has in turn, posed a variety of challenges to the scientific and engineering research community to provide the necessary algorithms and protocols to utilize these systems effectively and at times, overcome their drawbacks. Wireless networks such as satellite networks, radio networks, sensor networks, cellular networks, ad hoc networks and other mobile networks

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have become predominant because of the flexibility they offer. The applications of these networks include military communications, emergency systems and disaster recovery, e-commerce, etc. In this chapter, we survey only research that applies graph coloring, domination and clique problems in models of these networks. As with any modeling situation, these models capture only the basic aspects of the problem while several technical aspects are not captured in order to maintain the simplicity of the model. In many cases, even these "simple" models cannot be solved optimally for practical purposes and hence effort is made in this paper to identify heuristics, approximation algorithms and complexity results in this field of research.

The paper is organized as follows, Section 2 provides the notations and definitions of all the graph theoretic concepts used throughout this paper. In Section 3, we restrict our attention to the applications of graph domination problems to Mobile Ad Hoc Networks (MANET). Section 4 surveys the application of graph coloring and its variants that have been used to model channel assignment and scheduling type problems in wireless networks such as Packet Radio Networks and Cellular Networks. Cliques are most often not used in isolation for modeling, but as clusters in clustering problems or for deriving bounds for the graph coloring problems. Clique models are also utilized in traffic flow models, link interference models, etc. In Section 5, we deal with clique and independent set models in telecommunication in detail. Finally, we conclude with a summary in Section 6.

2 Notations and Definitions

Let $G = (V, E)$ be a simple loopless undirected graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E \subseteq V \times V$. Let n and m denote the number of vertices and edges, respectively. The *adjacency matrix* of the graph, A_G is the $n \times n$ matrix in which entry $a_{i,j} = 1$ if $(i, j) \in E$, and is 0 otherwise. For $i \in V$, let $N(i)$ denote the set of vertices adjacent to i called the *open neighborhood* and let $N[i] = N(i) \cup \{i\}$ denote the *closed neighborhood*. Let $deg_G(i) = |N(i)|$, denote the degree of vertex i in G. Denote by Δ and δ the maximum and minimum degrees respectively. The *closed neighborhood* of a set S is defined as $N[S] = \cup_{i \in S} N[i]$ and the *open neighborhood* of a set is defined as $N(S) = N[S] \setminus S$. The *complement graph* of G is the graph $\bar{G} = (V, \bar{E})$, where $\bar{E}$ is the complement of E. For a subset $W \subseteq V$ let $G(W)$ denote the subgraph induced by W on G which is obtained by deleting the set of vertices $V \setminus W$ and the incident edges from G. Also, by $d_G(i, j)$, denote the number of edges on the shortest path between vertices i and j in the graph G.

A dominating set $D \subseteq V$ is a set of vertices such that every vertex in the graph is either in this set or has a neighbor in this set. The minimum cardinality of a dominating set is called the *domination number*, denoted by $\gamma(G)$. A *connected dominating set* (CDS) is one in which the subgraph induced by the dominating set is connected. The *connected domination number* $\gamma_c(G)$ is the cardinality of the minimum connected dominating set (MCDS). The subgraph weakly induced by $D \subseteq V$ is the graph $G^w = (N[D], E^w)$, where E^w contains every edge in E that has at least one end point in D. The set D is a *weakly-connected dominating set* (WCDS) of G if D is dominating and G^w is connected. The minimum size of a WCDS is the *weakly connected domination number* of a graph, denoted by $\gamma_w(G)$. For a detailed description of notations and definitions relating to graph domination, refer [Hedetniemi and Laskar, 1997]. A set of vertices $I \subseteq V$ is called an independent set if for every $i, j \in I$, $(i, j) \notin E$, i.e. the graph $G(I)$ is edgeless. An independent set is *maximal* if it is not a subset of any larger independent set, and *maximum* if there are no larger independent sets in the graph. Note that a maximal independent set is also a minimal independent dominating set. The maximum cardinality of an independent set of G is called the *independence number* (or *stability number*) of the graph G and is denoted by $\alpha(G)$.

Generalizing the neighborhood definitions above to a distance k-neighborhood, leads to *k-domination* and *k-independence*. A subset $D \subseteq V$, is said to be a *k-dominating set* if every vertex not in D is reachable in no more than k steps from some vertex in D. Note that this is different from the k-domination defined in [Cockayne et al., 1985]. Vertices in $I \subseteq V$ are said to form a *k-independent set* if the length of the shortest path between any two vertices in the set is at least $k+1$. The size of the largest k-independent set in the graph is called the *k-independence number*, $\alpha_k(G)$. Note that $\alpha_1(G)$ is the same as $\alpha(G)$. $\gamma(G), \gamma_c(G)$ and $\alpha_k(G)$ are related as follows [Duchet and Meyniel, 1982].

$$\gamma(G) \leq \gamma_c(G) \leq 3\gamma(G) - 2$$

$$\gamma_c(G) \leq 2\alpha(G) - 1$$

$$\alpha_2(G) \leq \gamma(G) \leq \alpha(G)$$

The duality between $2k$-independence and k-domination for special classes of graphs is studied in [Chang and Nemhauser, 1984]. The NP-completeness of the decision version of these problems is also established here for bipartite and chordal graphs.

A *clique* C is a subset of V such that an edge exists between every pair vertices in C, i.e. subgraph $G(C)$ induced by C on G is complete. The maximum clique problem is to find a clique of maximum cardinality. The *clique number* $\omega(G)$ is the cardinality of a maximum clique in G. Note that $\omega(G) = \alpha(\bar{G})$. A proper coloring of a graph is one in which every vertex is colored such that no two vertices of the same color are adjacent. A graph is said to be *k-colorable* if it admits a proper coloring with k colors. Vertices of the same color are referred to as a *color class* and they induce an independent set. The *chromatic number* of the graph, denoted by $\chi(G)$ is the minimum number of colors required to properly color G. Note that for any graph G, $\omega(G) \leq \chi(G)$, as different colors are required to color the vertices of a clique. Analogously, in the *edge coloring problem* two edges that are incident at the same vertex have to be colored differently. The minimum number of colors required to color all the edges satisfying this condition is referred to as the *edge chromatic number* or the *chromatic index* denoted by $\chi'(G)$.

Computation of $\gamma(G)$, $\gamma_c(G)$, $\alpha(G)$, $\omega(G)$ and $\chi(G)$ for general graphs is difficult as the minimum dominating set, minimum connected dominating set, maximum independent set, maximum clique problems and the minimum vertex coloring problems are well known NP-hard problems [Garey and Johnson, 1979]. Minimum edge coloring has also been shown to be NP-hard [Hoyler, 1981] and hence finding $\chi'(G)$ is also difficult. Most of these problems remain NP-hard even on *Unit-Disk Graphs* (UDG) that are frequently used to model wireless networks [Clark et al., 1990]. Simple heuristics for these problems on UDG can be found in [Marathe et al., 1995]. It is also important to note that in these problems, the UDG model is given to us, as the problem of recognizing a UDG is also NP-complete [Breu and Kirkpatrick, 1993].

3 Graph Domination in Wireless Networks

MANET refers to distributed, wireless, multihop networks that function without using any infrastructure such as a base station or access points for communication. These are complex, dynamic systems owing to the mobility of nodes and hence lead to ad hoc network topologies.

Historically, ad hoc networks were utilized for military applications. Survival in a battlefield entails mobile wireless communication systems that can be used to coordinate and control operations on the battlefield in a distributed fashion without relying on centralized control stations that are prone to failure. But with the development of technologies such as Bluetooth and wireless internet, MANET have found applications in sensor networks, emergency services such as disaster recovery systems, in business environments, for conferencing and home networking, etc. In a recent work [Chlamtac et al., 2003], a comprehensive study of issues relevant to ad hoc networks and research activities in enabling technologies, networking protocols and services, etc., has been undertaken. Several books have been written recently [Perkins, 2001, Stojmenovic, 2002, Cheng et al., 2003a, Basagni et al., 2004] that provide information on research and development activities, issues and applications, with theory and relevant background material. Wireless ad hoc networks are networks which exhibit dynamic changes in their network topology. Clustering introduces a hierarchy that is other wise absent in these ad-hoc networks. Existence of a hierarchy facilitates routing of information through the network. Efficient resource management, routing and better throughput performance can be achieved through adaptive clustering of these mobile nodes.

Given the *connectivity graph* (also known as *communication graph, network graph*, etc.), $G^0 = (V^0, E^0)$, where the vertices represent the nodes in the network and the edges represent the communication links between pairs of nodes in the network, the *clustering problem* is to find subsets (not necessarily disjoint) $\{V_1^0, \ldots, V_k^0\}$ of V^0 such that $V^0 = \bigcup_{i=1}^{k} V_i^0$. Each subset is a cluster and it is required that the diameter of the subgraph induced by each subset is small. One vertex in each subset is distinguished as the *cluster-head* and the restriction on diameter ensures that every vertex in that subset is reachable in few steps from the cluster-head. After clustering, we can abstract the connectivity graph to a graph $G^1 = (V^1, E^1)$ as follows: there exists a vertex $v_i^1 \in V^1$ for every subset V_i^0 and there exists an edge between v_i^1, v_j^1 if and only if there exist $x^0 \in V_i^0$ and $y^0 \in V_j^0$ such that $(x^0, y^0) \in E^0$. We can recursively cluster the abstracted network to obtain a multi-level hierarchy.

The concept of graph domination has been frequently used for clustering. Once a dominating set is found, every vertex in it becomes a cluster-head and clusters are formed by the closed neighborhood of each vertex in the dominating set. The problem of finding a MCDS, which is frequently used in clustering is not only NP-hard, but also difficult to approximate [Lund and Yannakakis, 1994, Guha and Khuller, 1998] on general graphs. Hence several heuristics and approximation algorithms when the problem is restricted to certain special graph classes have been proposed to tackle this problem. For characterization and complexity classification of domination-type problems refer [Telle, 1994]. Another problem that is equivalent to finding a MCDS is finding a spanning tree of the graph with the maximum number of leaves which is also naturally NP-hard [Garey and Johnson, 1979]. The extremal aspects of these problems, bounds and heuristics are presented in [Caro et al., 2000]. In a recent unpublished work [Saxena, 2003], the author conducts a detailed study of polyhedral aspects of graph domination. This paper provides integer programming (IP) formulations of the dominating set problem and studies the characteristics of the associated polytope. Valid inequalities are provided and the relationships between the dominating set polytope, spanning tree polytope and matching polytope are studied. The following is the integer programming formulation for finding the domination number:

$$\gamma(G) = \min\{\mathbf{1}^T x : Ax \geq \mathbf{1},\ x \in \{0,1\}^n\} \tag{1}$$

Where $A = A_G + I$, A_G is the adjacency matrix of the graph, I is the $n \times n$ identity matrix and $\mathbf{1}$ is a $n \times 1$ vector of ones. Also recently, a fully distributed approximation algorithm has been developed, based on the linear programming (LP) relaxation of (1) in [Kuhn and Wattenhofer, 2003].

One of the earliest and simplest clustering algorithm for ad-hoc networks is the *linked cluster algorithm* found in [Baker and Ephremides, 1981] which is a distributed algorithm that produces a dominating set by scanning vertices in the decreasing order of their indices (also known as vertex IDs). They also suggest using some vertices outside the dominating set as "gateway nodes" in order to establish a backbone. In other words, they provide procedures that add more vertices to the dominating set that was originally found in order to obtain a connected dominating set. Similar algorithms based on indices and vertex degree are also proposed in [Gerla and Tsai, 1995], where an independent set of cluster-heads is found. It was also found that the vertex ID based algorithms are more robust in the face of changing network topology due to the mobility of nodes. However, these algorithms sometimes failed to produce dominating sets and some nodes were left without cluster-heads. This was resolved in a distributed linear-time *adaptive clustering algorithm* presented in [Lin and Gerla, 1991]. A more recent randomized distributed approximation algorithm that has time complexity $O(\log n \log \Delta)$ with high probability, for finding a dominating set with size no larger than $O(\log \Delta)$ of the optimal in expectation and $O(\log n \log \Delta)$ of the optimal with high probability can be found in [Jia et al., 2002].

In [Krishna et al., 1997], the authors define a *k-cluster* to be a subset of nodes that are mutually reachable by a path of length at most k. Since they deal only with 1-clusters (cliques) any further characterization was not necessary. Note that this definition allows 2 vertices to be in a cluster if the shortest path between them is of length at most k, even if the path uses vertices that are not in the cluster. Interestingly, this concept has been used in studying "cohesion" in social networks (also known as *acquaintance networks*) where it is referred to as *k-cliques*. An introduction to relevant definitions and basic properties of other clustering concepts such as *k-clubs* and *k-clans* that are used in studying social networks can be found in [Mokken, 1979]. Recently in [Balasundaram et al., 2005] these models were studied in the context of biological networks where some of the drawbacks in the definitions of these models were removed and NP-completeness results on arbitrary graphs as well as diameter-bounded graphs were presented. In addition, integer programming formulations for these problems were provided along with basic polyhedral properties of the 2-club problem. Note that, in social networks, where clustering is often accomplished by 2-clubs, the vertex set is partitioned into subsets (clusters) whose induced subgraphs have diameter at most two. Hence every vertex in the cluster is no more than two hops away from every other vertex. In other words, for every two vertices in the cluster there either exists an edge between them or a common neighbor *inside* the cluster. This essentially generalizes the notion of using a closed neighborhood to form a cluster by allowing subgraphs other than *stars* and thereby possibly avoiding the use of a centralized cluster-head for intra-cluster communication which could lead to congestion in the cluster. The *k-clustering* problem, as defined in [Deogun et al., 1997], is to find a partition of the vertex set of the graph such that the number of partite sets is a minimum and each partite set induces a subgraph whose diameter is bounded by k (k-clusters). This problem has been shown to be NP-hard and also hard to approximate in [Deogun et al., 1997]. The authors also present a polynomial-time approximation algorithm with a constant worst-case ratio of 3 for special classes of graphs that contain a *dominating diametral path* (DDP). The shortest path between two vertices in a graph is a DDP if its length is equal to the diameter of the graph and its vertices form a dominating set of the graph. Recently, a distributed polynomial-time approximation algorithm for this problem on unit disk graphs was presented in [Fernandess and Malkhi, 2002].

In [Basagni, 1999] a *distributed clustering algorithm* and a *distributed mobility-adaptive clustering algorithm* are presented that employ maximal weighted independent sets for clustering. A more recent distributed algorithm for clustering using maximal weighted independent sets can be found in [Basagni, 2001]. Usually, the weights are assigned to reflect the node's mobility, i.e. higher the mobility, lower the weights. This produces cluster-heads that are less mobile. In a similar approach, [Chatterjee et al., 2002] presents a *weighted clustering algorithm* that finds a dominating set based on weights assigned to every node. The weights are based on several system parameters like degree, distance, mobility and battery power.

Several distributed algorithms have been proposed [Sivakumar et al., 1998, Das and Bharghavan, 1997, Das et al., 1997] that are based on the algorithms presented in [Guha and Khuller, 1998] for MCDS. In [Guha and Khuller, 1998], the authors present two polynomial-time algorithms that have approximation ratios of $2H(\Delta) + 2$ and $H(\Delta) + 2$, where H is the Harmonic function given by $H(\Delta) = \sum_{i=1}^{\Delta} \frac{1}{i} \leq \ln \Delta + 1$. An algorithm with a time complexity of $O(\Delta^2)$ and a message complexity of $O(n\Delta)$ can be found in [Wu and Li, 2001]. Although they do not have any performance bounds, the authors find their algorithms to be more efficient compared to [Das et al., 1997] based on simulation results. An approximation algorithm for the MCDS problem on UDG with an approximation ratio of at most 8 is proposed in [Cardei et al., 2002]. This algorithm finds a maximal independent set and connects it using a Steiner tree in $O(n)$ time. A similar 8-approximate algorithm that runs in linear time is presented in [Butenko et al., 2003]. Here again, a maximal independent set is found in the first phase and is connected using a tree in the second phase. Both these algorithms have a message complexity of $O(n\Delta)$. Other approximation algorithms for MCDS in UDG have been proposed in [Alzoubi et al., 2002a, Wan et al., 2004] that have time complexity of $O(n)$, message complexity of $O(n \log n)$ and an approximation factor of 8. A lower bound of $\Omega(n \log n)$ on message complexity was also established in [Wan et al., 2004] showing their algorithm to be message optimal. The same authors also present an algorithm in [Alzoubi et al., 2002b] with linear time and message complexity where each message is $O(\log n)$ bits long. A heuristic for MCDS on a restricted class of graphs that do not contain cliques as minors is presented in [Wan et al., 2003]. The approximation ratio depends on the clique size that's forbidden and for planar graphs in particular, this heuristic produces a connected dominating set of size at most $15\alpha_2(G) - 5$. Recently, a polynomial time approximation scheme for the MCDS problem on UDG have been proposed in [Cheng et al., 2003b]. A heuristic approach to the MCDS problem that is different from other existing constructive algorithms has recently been proposed in [Butenko et al., 2004a]. The authors present a $O(mn)$ time complexity, vertex elimination type algorithm that starts with the entire vertex set as the solution and proceeds by recursively removing a vertex of minimum degree, at the same time ensuring connectedness and domination property (thereby having a feasible solution at all stages of the algorithm). Two distributed algorithms for finding connected dominating sets in unit disk graphs, that break the linear-time barrier have been proposed very recently in [Parthasarathy and Gandhi, 2004]. These algorithms are based on the distance-2 coloring of the ad hoc network.

The use of WCDS has also been proposed recently for clustering ad hoc networks [Chen and Liestman, 2002]. The problem of finding a WCDS has been shown to be NP-hard in [Dunbar et al., 1997], where a detailed study of this problem is carried out. Sharp upper and lower bounds on $\gamma_w(G)$ and its relation to other domination parameters are also provided in the same paper. Approximation algorithms based on algorithms presented in [Guha and Khuller, 1998] have been proposed recently in [Chen and Liestman, 2002] that have an approximation ratio of $O(\ln \Delta)$. A *zonal algorithm* for finding weakly-connected dominating sets is presented in [Chen and Liestman, 2003]. Distributed polylogarithmic-time approximation algorithms for finding connected and weakly-connected dominating sets has recently been proposed in [Dubhashi et al., 2003] with an approximation ratio of $O(\log \Delta)$. In [Alzoubi et al., 2003b], two linear time algorithms are presented, one of which has an approximation ratio of 5 and a message complexity of $O(n \log n)$ and the other has a higher approximation ratio but a linear message complexity. Randomized greedy algorithms for finding WCDS in regular graphs is presented in [Duckworth and Mans, 2003].

A different approach to clustering involves the use of distance k-neighborhood, where all nodes that are at a distance of no more than k from the cluster-head are included in the cluster. In other words, the set of cluster heads forms a *k-dominating set*. The problem of finding a minimum k-dominating set in UDG is shown to be NP-hard in [Amis et al., 2000]. The authors also propose a heuristic to solve this problem. Such clustering algorithms, based primarily on degree (and vertex IDs for tie-breaking) can be found in [Nocetti et al., 2003]. A k-dominating set is said to be small if it has no more than $\max\{1, \lfloor \frac{n}{k+1} \rfloor\}$ vertices in it. Distributed algorithms of time complexity $O(k \log^* n)$ for finding small k-dominating sets can be found in [Kutten and Peleg, 1998, Penso and Barbosa, 2004], where $\log^* n = \min\{i : \log^{(i)} n \leq 2\}$ and $\log^{(i)} n = \log \log^{(i-1)} n$ with $\log^{(1)} n = \log n$.

Other approaches include mobility-based clustering designed to capture common mobility characteristics among nodes [An and Papavassiliou, 2001, Sivavakeesar and Pavlou, 2002], access-based clustering reported in [Hou and Tsai, 2001] and a recent clustering approach proposed in [Bannerjee and Khuller, 2001] where explicit constraints on the size, connectivity and overlap of clusters is enforced. The notion of dominating sets is extended to dominating and absorbant sets in directed graphs in [Wu, 2002] and algorithms are provided that are shown to be effective by simulations. In [Huang et al., 2004], the authors introduce a *Mobile Piercing Set Problem* which is similar to the dominating set problem on UDG. They provide distributed approximation algorithms for this problem on UDG, as well as non-uniform disks (disks of different radii). Besides clustering, graph domination has been used to model other problems that arise in ad hoc networks as well. Broadcasting algorithms that utilize dominating sets can be found in [Stojmenovic et al., 2002, Wu et al., 2003, Ingelrest et al., 2004].

In this section, we have tried to capture the research activity in ad hoc networks with regards to graph domination. A literature review and references to some recent developments in clustering algorithms for wireless networks have been presented. Recent papers that deal with theoretical and algorithmic complexity aspects of graph domination have also been referred. Extensive study of domination in graphs has been carried out in [Haynes et al., 1998a,b]. These books cover the basics of graph domination, as well as advanced topics and recent research activities in this field. A comprehensive bibliography is also provided. A recent survey that focusses on graph domination and clustering in ad hoc networks can be found in [Chen et al., 2004] and provides more details regarding other technical aspects of this problem as well. Also, for a recent detailed study of impacts of clustering in ad hoc networks and a comparison of different mechanisms used based on various metrics, refer [Hossain et al., 2004].

4 Graph Coloring in Wireless Networks

The graph coloring problem on general graphs is a well known NP-complete problem and it is hard to approximate [Garey and Johnson, 1979, Lund and Yannakakis, 1994]. Even on UDG, this problem is found to be NP-hard [Clark et al., 1990, Gräf et al., 1998] and a 3-approximation algorithm for coloring UDG is known [Gräf et al., 1998]. However, the maximum clique problem can be solved in polynomial time on UDG [Clark et al., 1990] and often heuristics for the maximum clique problem are utilized to derive lower bounds on the chromatic number of the graph.

The graph coloring problem is often used to model channel assignment type problems such as code assignment, frequency assignment and time-slot assignment in wireless networks. Protocols such as FDMA (Frequency Division Multiple Access), TDMA (Time Division Multiple Access), CDMA (Code Division Multiple Access), etc., are designed to limit the number of users sharing the channel simultaneously. Despite significant differences from practical implementation point of view, FDMA protocol is not very different from a TDMA protocol from an algorithmic point of view, as the same conditions are necessary for the transmissions to be collision-free. In the frequency domain however, the total bandwidth is divided into frequency bands, which are equivalent to the time-slots that are assigned in the time domain. So, the heuristics available for a TDMA framework apply for FDMA as well. In fact, most papers surveyed do not restrict the application of their models and heuristics to the TDMA networks alone, but also to the FDMA networks. However, some papers specifically address the frequency assignment problem with different models that are variants of the basic graph coloring problem. These will be dealt with separately towards the end.

In multihop wireless networks, two nodes that are not directly connected, communicate by sending information packets through intermediate nodes. Under these circumstances, *collision* is said to have occurred if any node receives packets from more than one node at the same time. Such collisions lead to a wastage of resources in terms of delay and bandwidth used to retransmit the same information. Two types of collisions, *primary or direct collisions* and *secondary or hidden collisions* are modeled using graph coloring. The former are collisions that occur between nodes that are within the transmission range of each other and the latter are collisions that occur at a node due to simultaneous transmissions from nodes that are themselves not in the hearing range of each other. Scheduling problems that arise in a TDMA framework are *link or singlecast scheduling* and *broadcast scheduling*. The fundamental difference being, in link scheduling, the transmission from a node is intended for one node only whereas in broadcast scheduling, the transmission from one node is intended for all the nodes in its neighborhood. Note that both these types are special cases of *multicast scheduling* where a node's transmission is destined for a subset of stations in its neighborhood. In this section, we look at applications of graph coloring problem and its variants to channel assignment problems (time-slot/frequency/code) and scheduling problems in wireless networks.

The greedy coloring algorithm is a simple and very popular heuristic for graph coloring. This algorithm proceeds by ordering the vertices and coloring them in that order with the smallest color (the colors are usually identified by an integer) not yet assigned to any of its neighbors that appeared before in the ordering. Since, each vertex can have at most Δ neighbors before it in the ordering, this algorithm will require no more than $\Delta+1$ colors to color the graph and thus we have $\chi(G) \leq \Delta + 1$. As we will see very soon, this heuristic is used often in developing algorithms for scheduling problems in wireless networks. Note that this bound is sharp for complete graphs and odd cycles. It is sharp in fact only for these graphs as it has been established in [Brooks, 1941] that $\chi(G) \leq \Delta$ unless the graph is complete or an odd cycle. It should be noted that there always exists an ordering for which greedy algorithm is optimal, i.e. the vertices can be ordered in the non-decreasing order of colors assigned to them in an optimal coloring and the greedy algorithm will produce optimal coloring in this case.

Note that the link scheduling problem can be modeled as an edge coloring problem and broadcast scheduling problem as vertex coloring problem. Let $G = (V, E)$ represent the connectivity graph of the wireless network. Construct $G^2 = (V, E^2)$ where $(i, j) \in E^2 \Leftrightarrow d_G(i, j) \leq 2$. Graph coloring on G and G^2 model direct collisions and both direct and hidden collisions, respectively. Note that coloring G^2 is the same as the *distance 2-coloring* of G. In distance k-coloring, two vertices i, j can have the same color if and only if $d_G(i, j) \geq k+1$. The distance k-coloring problem was shown to be NP-complete for any fixed $k \geq 2$ in [McCormick, 1983] in the context of approximating sparse Hessian matrices. An $O(\sqrt{n})$-approximation for distance 2-coloring was also presented in the same paper.

The graph G^2 is called the square of G. In general, the k^{th} power of a graph G, denoted by G^k, is a graph $G^k = (V, E^k)$ where $(i, j) \in E^k \Leftrightarrow d_G(i, j) \leq k$. Note that it is a well known result [Appel and Haken, 1977, Appel et al., 1977, Robertson et al., 1997] that for a (loopless) planar graph G, $\chi(G) \leq 4$. Let Δ denote the maximum degree in G which is planar, some recent results of interest are $\chi(G^2) \leq 2\Delta + 25$ proven in [van den Heuvel and McGuinness, 2003] and $\chi(G^2) \leq 3\Delta + 9$ proven in [Jendrol' and Skupień, 2001]. It can also be easily seen that $\chi(G^2) \geq \omega(G^2) \geq \Delta + 1$ as a vertex with the maximum degree in G, along with its neighbors would form clique in G^2. For general powers of graphs, a recent work [Agnarsson and Halldórsson, 2003] derives bounds on the *inductiveness* of G^k when G is planar, where inductiveness of a graph G is defined as $\max_{S \subseteq V(G)} \{\min_{v \in S} deg_{G(S)}(v)\}$. This can be used to derive bounds on the chromatic number and for developing algorithms as it leads to ordering of vertices for the greedy coloring algorithm.

If the transmission radius is not the same for every node, directed graphs need to be used as bidirectional nature of communications cannot be assumed. Then the UDG has to be generalized to a digraph $G = (V, A)$ that has vertices representing the radio stations and a directed edge from u to v if and only if v can receive u's transmission. The problem of broadcast scheduling reduces to a vertex coloring problem on graph G. Two vertices u and v can be colored the same if and only if $u \rightarrow v \notin A$ and $v \rightarrow u \notin A$ and there doesn't exist a w such that $u \rightarrow w \in A$ and $v \rightarrow w \in A$, thereby avoiding primary and secondary collisions. The link scheduling problem reduces to an edge coloring problem where two edges $u \rightarrow v$ and $x \rightarrow w$ can be assigned the same colors if and only if u, v, x, w are all distinct, the edges $u \rightarrow w$ and $x \rightarrow v$ do not exist.

The NP-hardness of various forms of the link and broadcast scheduling problems has been long since established. In [Even et al., 1984], complexity results of network testing problems are established, some of which are close to scheduling problems that are under consideration. Link scheduling problems to avoid primary and secondary collisions in packet radio networks are modeled as digraphs in [Arikan, 1984], where the problem of deciding whether a given origin-to-destination message rates are achievable via any arbitrary protocol is shown to be NP-complete. However, the link scheduling to meet link demands and end-to-end demands was shown to be polynomial time solvable in a spread spectrum framework where secondary collisions are tolerated in [Hajek and Sasaki, 1988]. This algorithm was later improved in [Ogier, 1986]. Some algorithms for link scheduling can be found in [Chlamtac and Lerner, 1985, Hajek and Sasaki, 1988]. Similar to feasibility results obtained in [Arikan, 1984] for link scheduling, the feasibility problem for broadcast scheduling, where secondary collisions are permitted, is shown to be NP-complete and several heuristic algorithms for the problem are presented in [Bonuccelli and Leonardi, 1997]. The problem of finding the largest broadcasting set, i.e. the largest set of vertices that can broadcast simultaneously (the largest 2-independent set on arbitrary undirected graphs) is shown to be NP-hard in [Ephremides and Truong, 1990, Ramaswami and Parhi, 1989]. In addition, NP-hardness of minimum length schedule for broadcast scheduling (distance 2-coloring on arbitrary undirected graphs) is proven in [Ramaswami and Parhi, 1989]. Centralized and distributed algorithms based on the greedy coloring heuristic for the minimum length scheduling problem are presented in [Ramaswami and Parhi, 1989] as well. Distributed algorithms for broadcast scheduling can also be found in [Cidon and Sidi, 1989, Ephremides and Truong, 1990]. The NP-hardness of distance-2-coloring of planar graphs, as well as approximation algorithms for the planar and general graphs are presented in [Lloyd and Ramanathan, 1992]. A *phase allocation* algorithm for channel sharing in wireless multihop networks is presented in [Chlamtac and Kutten, 1985b] and a *distributed link allocation algorithm* is presented in [Chlamtac and Lerner, 1985] . The time-slot assignment problem is modeled on a digraph and NP-hardness of minimizing the maximum time and the average time (over all nodes) needed for the broadcasted message to reach all nodes are established and a distributed heuristic based on the greedy coloring algorithm are presented in [Chlamtac and Kutten, 1985a]. A distributed version of this algorithm is used to solve the time slot assignment problem in [Chlamtac and Pinter, 1987]. The digraph model is used in [Ramanathan, 1993] to study scheduling problems. Given the hardness of these problems in general graphs, the complexity aspect of these problems restricted to trees and planar graphs is studied in [Ramanathan, 1993]. Although the problems are polynomial for trees, they remain NP-hard for planar graphs. Approximation algorithms for general graphs based on the thickness of the graphs are also presented. The minimum number of planar graphs into which a given graph can be partitioned is called the thickness of the graph (θ). An $O(\theta^2)$ approximation algorithm for link scheduling that runs in $O(mn \log n + n\theta^2\Delta)$ time and an $O(\theta)$ approximation algorithm for broadcast scheduling with time complexity $O(n\theta\Delta)$ are presented in [Ramanathan, 1993]. Similarly in [Sen and Huson, 1997], the digraph models (referred to as *point graphs*) are used and the NP-completeness of distance 2-coloring of a much restricted class of planar point graphs is established. Approximation algorithms for special cases are also provided. Recently, distance 2-coloring on undirected graphs has been studied in [Krumke et al., 2001] where approximation algorithms for the distance 2-coloring problem on special graphs used to model packet radio networks are presented and a 2-approximate algorithm for this problem on bounded degree planar graphs is also presented. A simulated annealing type heuristic is applied to the broadcast scheduling problem in [Wang and Ansari, 1997] and is found to be effective. A two-phased algorithm based on sequential vertex coloring is presented in [Yeo et al., 2002] where the length of the schedule is minimized in phase one and utilization is maximized in phase two. In a similar two phase heuristic is presented in [Salcedo-Sanz et al., 2003] where the first phase uses a Hopfield neural networks [Hopfield, 1982, Hopfield and Tank, 1985] for minimizing schedule length and genetic algorithms(similar to [Watanabe et al., 1998]) to maximize slot utilization. A Greedy Randomized Adaptive Search Procedure (GRASP) [Feo and Resende, 1995, Resende and Ribeiro, 2003, 2005] is developed for the broadcast scheduling problem in [Butenko et al., 2004b]. GRASP is a two-phase meta-heuristic that has been used quite successfully in the recent past for various combinatorial optimization problems [Festa and Resende, 2001]. In a recent paper [Commander et al., 2004] a comparative study of these four metaheuristic approaches is performed and all the heuristics are found to be competitive, with GRASP finding shorter schedules than the other heuristics in all the randomly generated test cases. In [Laguna and Martí, 2001], GRASP for coloring sparse graphs is presented with an extensive survey of literature on heuristics applied to coloring problems.

The frequency assignment problem has been researched extensively, especially because of its application in cellular networks, and the trade-offs between bandwidth usage and system interference studied. As suggested in [Hale, 1980], besides minimizing the length of the schedule, referred to as *order* in this context, it is also important to minimize *span*, which is the difference between the largest and the smallest frequencies assigned. The author also establishes some fundamental results in this respect. This work is further extended in [Cozzens and Roberts, 1982] where T-colorings of graphs and multigraphs are used to model the channel assignment problem. In T-colorings, given a set of non-negative integers T, a T-coloring on graph G 'colors' every vertex by assigning an integer from T to every vertex such that the difference of colors on any two adjacent vertices does not fall in T. A recent survey of the frequency assignment problem, detailing several technical aspects and a host of available solution methods and heuristics including graph coloring heuristics are presented in [Murphey et al., 1999]. Another variation that is also applicable to frequency assignment problems is called $L(p, q) - labeling$. A labeling of a graph $\varphi : V \longrightarrow \{0, \ldots, k\}$, for given integers $p, q, k \geq 0$, is called an $L(p, q) - labeling$ if it satisfies:

$$|\varphi(u) - \varphi(v)| \geq p \ \ \forall u, v \ such\ that\ d_G(u, v) = 1$$

$$|\varphi(u) - \varphi(v)| \geq q \ \ \forall u, v \ such\ that\ d_G(u, v) = 2$$

The $p, q - span$ of a graph G denoted by $\lambda(G; p, q)$ is the minimum k for which an $L(p, q) - labeling$ exists. Note that $\lambda(G; 1, 0) = \chi(G)$. If G is planar, then $\lambda(G; p, q) \leq (4q-2)\Delta + 10p + 38q - 24$ [van den Heuvel and McGuinness, 2003]. Other results of interest and relevant references can be found in [van den Heuvel and McGuinness, 2003].

CDMA protocols are widely used in cellular networks, wireless local area networks and wireless ad hoc networks. CDMA protocols provide higher capacity, flexibility, scalability, reliability and security than conventional FDMA and TDMA [Wan, 2004]. They enable proper channel sharing by the assignment of *orthogonal codes* which are nothing but pseudo-random binary codes. The data that is transmitted is duplicated and multiplied by these codes before transmission and the entire spectrum is used to communicate. The number of duplicates is equal to the length of the code word and is known as the *spreading factor*. The longer the code, more robust is the communication but at a lower rate as rate is defined as inverse of code length.

When all the codes have a fixed code length as in conventional CDMA, it is called OFSF-CDMA (for Orthogonal Fixed Spread Factor - CDMA). In this case, the code assignment problem is combinatorially identical to the previously seen TDMA and FDMA assignment problems. Naturally, the heuristics and algorithms for this problem have also been developed in a similar way. Centralized and distributed heuristics for the hidden interference problem are presented in [Bertossi and Bonuccelli, 1995]. This work is further extended in [Battiti et al., 1999] where a *saturation degree code assignment heuristic* is presented for the graph coloring problem which is based on [Brélaz, 1979]. Here, the vertex with the highest number of colors in its neighborhood is colored first.

However, recently variable length codes are being used in a CDMA framework, referred to as the OVSF-CDMA (orthogonal variable spreading factor - CDMA). Here the codes are represented by a binary tree structure. Under this situation, the primary and hidden collisions need to be redefined as well as the graph model used to capture this problem. This is done using *prefix free vertex coloring*. The readers are referred to [Wan, 2004, Minn and Siu, 2000] for details regarding definitions, results, heuristics and assignment schemes used for solving these problems.

In [Ramanathan, 1999], a unified framework is presented for studying the assignment problems and algorithms for TDMA/FDMA/CDMA channel assignments are presented based on this framework. For a discussion on the various technical aspects of channel assignment problem and for a survey of assignment schemes, refer [Katzela and Naghshineh, 1996].

5 Cliques in Wireless Networks

The maximum clique problem is a well known NP-hard problem [Garey and Johnson, 1979] that is also hard to approximate [Håstad, 1999]. The maximum independent set problem is closely related to the maximum clique problem since a set C is a clique in G if and only if it is an independent set in its complement, $\bar{G}$. Hence, we will discuss the applications of both these graph models in telecommunications in this section. An extensive survey of the maximum clique problem can be found in [Bomze et al., 1999]. This contains pointers to several fundamental formulations and results besides presenting a host of available heuristics and solution techniques.

As a maximal independent set is also a dominating set, it is often used in clustering wireless networks. In fact, some of the earliest clustering algorithms find maximal independent sets and use them as dominating sets for clustering. Maximal independent sets are specially favored in clustering as they provide cluster-heads with desirable domination and independence properties. Effective algorithms are also known for finding maximal independent sets in general graphs. Probabilistic parallel algorithms for the problem are presented in [Luby, 1986, Alon et al., 1986, Goldberg and Spencer, 1989]. As mentioned in Section 3, in [Basagni, 2001] the author presents a distributed algorithm for finding maximal weighted independent sets in graphs. It is also shown using results from the theory of *random graphs*, that the average time complexity of this algorithm is $O(\log_b n)$, where b indicates the probability of having an edge between any pair of nodes in the network. Some fundamental properties of this problem on UDG and algorithms for clustering based on the construction of maximal independent sets are presented in [Alzoubi et al., 2003a]. More recently a polylogarithmic algorithm has been proposed in [Moscibroda and Wattenhofer, 2004] for solving maximal independent set problem on UDG.

Another modeling approach is to partition the graph into clusters where each cluster is a clique. This has the advantage that control in each cluster is not centralized and each node is one hop away from every other node in the cluster. The problem of partitioning a graph into cliques is NP-complete [Garey and Johnson, 1979]. However, recent results in [Krishnamachari et al., 2003] regarding phase transition in the complexity of this problem with transmission radius on random graphs provide some useful insights into the practical issues involved in solving this problem. Similar results are also obtained for channel assignment problems, i.e. graph coloring and distance-2 coloring.

As the maximum clique problem is polynomial time solvable in UDG, exact algorithms or efficient heuristics are often used to find $\omega(G)$ or a lower bound on $\omega(G)$ which is used as lower bound for the chromatic number of the graph. As mentioned before in Section 3, this approach is used in [Battiti et al., 1999] to study the bounds and scaling properties of the code assignment problem in CDMA framework. They use a Reactive Local Search (RLS) heuristic for maximum clique problem presented in [Battiti and Protasi, 2001] for deriving bounds on the chromatic number using the best clique size found. RLS heuristic is a $O(\max\{n, m\})$ running time heuristic that has been shown to produce good computational results on the DIMACS benchmark problems [DIMACS, 1995]. It complements local search by a *feedback* scheme used to control *prohibition-based diversification.* Prohibition refers to the fact that some local neighbors are forbidden from being visited in order to explore new regions of the global solution space. Feedback is an internal way of maintaining history of solutions visited in order to automate tuning of the algorithm as it is solving a particular instance. These principles are similar to Glover's *Tabu Search* algorithm [Glover and Laguna, 1997, Glover, 1989, 1990].

The maximum independent set problem on a connectivity graph, yields the maximum broadcasting set in a TDMA framework as presented in [Ramaswami and Parhi, 1989] or an activation set in a multihop radio network as suggested in [Tassiulas and Ephremides, 1992]. Fundamentally, independent set models are used to identify mutually non-conflicting nodes in the network, with respect to some shared resource. In addition, if secondary conflicts such as hidden collisions defined in Section 3 are to be avoided, then the problem reduces to finding the largest 2-independent set. The maximum independent set is inherently a hard problem to solve. Another dimension is added to the complexity of this problem in the present context where this problem has to be solved often to keep up with the dynamic network topology. This is suggested by the NP-hardness of the problem of adapting a maximum broadcasting set to a simple topology change of adding nodes to the network [Vuong and Huynh, 1999]. But this is not surprising given that the difficulty in solving this and other such problems arises from the fact that given a vertex it is hard to determine whether or not it belongs to a maximum independent set, under nontrivial circumstances.

In modeling interference between wireless links in a network, a *conflict graph* (also referred to as *contention graph, interference graph*, etc.) is often used. A conflict graph $G^c = (V^c, E^c)$ has a vertex for every edge in the connectivity graph ($V^c = E$). An edge exists between two vertices in G^c if the corresponding links in G interfere. Usually, two links in G are said to interfere if their mid-points lie within the interference radius of each other.

The authors of [Gupta and Walrand, 2004] utilize maximal cliques to model interference between the links (edges) in the connectivity graph ($G = (V, E)$) of a wireless network. A clique in G^c is set of links such that at most one link from that set can be active. In other words, no two links in a clique can be active simultaneously if link interference has to be avoided. The problem that is addressed in this work is to find, given a link $l \in V^c$, all maximal cliques that contain this link. The *scanning disk heuristic* proposed in this paper is a fully distributed heuristic to find all maximal cliques containing a given vertex in UDG. Simulation results indicate that the heuristic is fast on the random UDG tested.

In [Puri, 2002] the problem of maximizing traffic flow in a fixed wired and wireless network is studied. The wireless links can interfere and with these wireless links as vertices, a conflict graph is constructed. Time division multiplexing is assumed and a schedule dictates during which time slot each link is active. Objective is to maximize revenue which is generated according to the amount of flow between every origin-destination pair in the wireless network subject to capacity constraints on individual links. The problem of finding a feasible schedule that maximizes revenue is shown to be NP-hard. A linear programming formulation that utilizes a clique-constraint for every clique in the graph is used to obtain an upper bound on the optimum revenue and a heuristic is presented for finding good feasible schedules. Alternately, use of independent sets is also suggested and corresponding formulations are presented. In a similar approach, [Montemanni et al., 2001] uses clique constraints in order to strengthen the linear programming relaxation of the integer formulation of the frequency assignment problem to obtain good bounds on the optimum. In [Kalvenes et al., 2005], a similar approach is used for a different IP formulation. In this paper, a revenue maximization model is proposed that utilizes clique cuts that are added successively by solving maximal clique problems on the interference graph. Elaborate preprocessing and other techniques are proposed to improve the effectiveness of the LP relaxation in obtaining good solutions in reasonable amounts of time. Computational results are found to be very encouraging with this approach. The problem of resource allocation in a wireless multihop ad hoc networks is studied in [Xue et al., 2003]. The objective is to maximize the aggregated utility over the network subject to bandwidth availability to competing multihop flows. A pricing approach is used to decide resource allocation. They propose the use of a link contention graph and shadow prices are associated with maximal cliques in this graph. A distributed algorithm is presented that determines shadow prices such that aggregated utility of all flows is maximized. The mathematical program is a non-linear model that is handled using Lagrangian methods and the associated Lagrange multipliers give rise to shadow price associated with the corresponding maximal clique. More recently in [Fang and Bensaou, 2004], a game theoretic model has been developed for fair-sharing of bandwidth in wireless ad-hoc networks. A primal problem which is a constrained maximization problem is developed and Lagrangian relaxation and duality theory are used to determine the game formulation of the problem. Algorithms for solving these problems are also provided.

Before concluding this section, drifting away from wireless networks, we look at an application of clique model to a different telecommunication problem. This model is interesting for several important reasons. The graph under consideration does not "directly" come from a physical network, it is very different from the applications we restricted our attention to in this section and the graphs under consideration are massive. The graphs we are dealing with are called *Call graphs* whose vertices are telephone numbers, and two vertices are connected by an edge if a call was made from one number to another. Such massive graphs representing telecommunications traffic data are presented in [Abello et al., 1999]. A call graph based on one 20-day period had 290 million vertices and 4 billion edges. The analyzed one-day call graph had 53,767,087 vertices and over 170 millions of edges. This graph appeared to have 3,667,448 connected components, most of them tiny; only 302,468 (or 8%) components had more than 3 vertices. A giant connected component with 44,989,297 vertices was computed. It was observed that the existence of a giant component resembles a behavior suggested by the random graphs theory of Erdös and Rényi, but by the pattern of connections the call graph obviously does not fit into this theory. The maximum clique problem and problem of finding large quasi-cliques with prespecified edge density were considered in this giant component. These problems were attacked using a greedy randomized adaptive search procedure (GRASP) [Feo and Resende, 1995, 1994]. To make application of optimization algorithms in the considered large component possible, the authors use some suitable graph decomposition techniques employing external memory algorithms. 100,000 GRASP iterations were needed, taking 10 parallel processors about one and a half days to finish. Of the 100,000 cliques generated, 14,141 appeared to be distinct, although many of them had vertices in common. The authors suggested that the graph contains no clique of a size greater than 32. Finally, large quasi-cliques with different density parameters were computed for the giant connected component. Approaches to detect quasi-cliques are detailed in a recent work presented in [Abello et al., 2002].

6 Conclusion

In this paper, we have attempted to provide a detailed survey of the use of graph models in telecommunications. Given the vast amount of literature generated in this line of research, it is almost impossible to present a comprehensive survey of all available literature. Hence we restrict our attention to wireless networks and identify research that elucidates the idea behind selected graph models. Our main objective was to strike a balance between the breadth of literature covered and the depth to which individual works are presented. Although we restrict our attention mostly to wireless networks, the principles and the modeling criteria are often the same wherever these graph optimization problems are used. Besides providing the models and solution methods that exist in telecommunication literature, we also present recent theoretical developments in graph theory and complexity theory with regards to these graph models that are of interest and relevance.

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