

On minimum connected dominating set problem in unit-ball graphs

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Abstract

Given a graph, the minimum connected dominating set problem is to find a minimum cardinality subset of vertices D such that its induced subgraph is connected and each vertex outside D has at least one neighbor in D . Approximations of minimum connected dominating sets are often used to represent a *virtual routing backbone* in wireless networks. This paper proposes a constant-ratio approximation algorithm for the minimum connected dominating set problem in unit-ball graphs.

Key words: Minimum connected dominating set, approximation algorithm, discrete optimization, wireless networks

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1 Introduction

Consider a simple undirected graph $G = (V, E)$ with the set of vertices V and the set of edges E . For a subset W of vertices, we will denote by $G[W]$ the subgraph induced by W , *i.e.*, $G[W] = (W, E \cap (W \times W))$. For a vertex $i \in V$, $N(i) = \{j \in V : (i, j) \in E\}$ is the neighborhood of i and $\deg(i) = |N(i)|$ is the degree of vertex i . For $S \subseteq V$, let $N^\cup(S) = \{v \in V \setminus S \mid \exists u \in S : (u, v) \in E\}$. A subset I of vertices is called an independent set if $G[I]$ has no edges. It is called a maximal independent set if it is not a subset of a larger independent set. $D \subseteq V$ is called a dominating set of G if each vertex $v \in V$ is either in D or has at least one neighbor in D . For a connected graph G , a dominating set D is called a connected dominating set if it induces a connected subgraph $G[D]$ in G . The cardinality of a minimum connected dominating set in G is called the connected domination number of G and is denoted by $\gamma_c(G)$. Note that the problem of finding a minimum connected dominating set in a graph is equivalent to the problem of finding a spanning tree with maximum number of leaves in G , with all non-leaf nodes in such a spanning tree forming a minimum connected dominating set.

In this paper we are interested in computing small connected dominating sets for a restricted class of graphs called unit-ball graphs. A graph G is called a unit-ball graph if its vertices can be represented as points in 3-dimensional Euclidean space \mathbb{R}^3 so that the distance between two points corresponding

to an edge in G is less than 1. Geometrically, this means that two points u and v in \mathbb{R}^3 are connected by an edge if u is inside the ball of radius 1 centered in v . Unit-ball graphs are gaining popularity in modeling ad hoc wireless networks, where transmitting hosts are represented by dots in \mathbb{R}^3 and the unit length represents the host's transmission range (which is assumed to be the same for all hosts). Traditionally, such networks were modeled using unit-disk graphs, a 2-dimensional analog of the unit-ball graphs studied in this paper, however unit-ball graphs appear to provide a more reasonable representation of a real-life wireless network. Since the unit-disk graphs can be viewed as a subclass of the unit-ball graphs, in which all dots are restricted to be coplanar, all the negative complexity results previously obtained for unit-disk graphs remain valid for the class of unit-ball graphs. In particular, it is known that recognizing unit-disk graphs is NP-hard. In addition, the maximum independent set, minimum vertex cover, minimum dominating set, minimum independent dominating set, and minimum connected dominating set problems are known to be NP-hard for unit-disk graphs [2], and therefore are all NP-hard for unit-ball graphs as well.

For a minimization problem \mathcal{P} , the *performance ratio* of an approximation algorithm \mathcal{A} is defined as

$$\rho(\mathcal{A}) = \sup_{\pi \in \mathcal{P}} \frac{\mathcal{A}_\pi}{opt_\pi},$$

where π is an instance of \mathcal{P} , \mathcal{A}_π is the output of \mathcal{A} on instance π , and opt_π is the optimal solution value for instance π . There are several polynomial-time

algorithms with performance guarantee for the minimum connected dominating set problem. In particular, Guha and Khuller [4] propose an algorithm which gives a performance ratio of $\ln \Delta + 3$, where Δ is the maximum degree of the graph. A Polynomial Time Approximation Scheme (PTAS) for the minimum connected dominating set problem in unit-disk graphs is also possible, as shown in [5] and more recently in [1].

The remainder of this paper is organized as follows. Section 2 establishes a relation between the sizes of a maximal independent set and a minimum connected dominating set in a unit-ball graph. Our algorithm for the minimum connected dominating set problem and its performance analysis are presented in Section 3. Section 4 concludes the paper.

2 Preliminaries

Consider an equivalent geometric representation of a unit-ball graph, in which instead of the balls of radius 1 we use the balls of diameter 1 centered in each point representing a vertex of G . Then $(u, v) \in E$ if and only if the open balls of diameter 1 centered in u and v have a nonempty intersection (here and henceforth we will use the same notation u for a point in \mathbb{R}^3 representing the vertex u of G).

Lemma 2.1 *Given an arbitrary vertex v of a unit-ball graph G , its neighborhood $N(v)$ cannot contain an independent set of size greater than 12.*

Proof: We will use the geometric concept of *kissing number*, which is the maximum number of spheres of radius 1 that can simultaneously touch the unit sphere in n -dimensional Euclidean space. For $n = 3$, the kissing number is known to be equal to 12 [3], which implies the lemma's statement. \square

Lemma 2.2 *The size of any maximal independent set I of a unit-ball graph G is at most $11\gamma_c(G) + 1$. Moreover, if $\max\{|N(v) \cap I| : v \in V \setminus I\} \leq \nu \leq 11$ then $|I| \leq \nu\gamma_c(G)$.*

Proof: Let $I = \{u_1, \dots, u_p\}$ be an arbitrary maximal independent set and let $D = \{v_1, \dots, v_q\}$, where $q = \gamma_c(G)$ be a minimum connected dominating set in G . Without loss of generality, we can assume that the vertices in D are listed in the order of an arbitrary depth-first search performed on $G[D]$. Let us partition the vertices in I into disjoint subsets I_1, \dots, I_q so that each vertex in $I_j, j = 1, \dots, q$, is a neighbor of v_j , but is not a neighbor of any of the preceding vertices v_1, \dots, v_{j-1} in D . Then due to Lemma 2.1, $|I_1| \leq 12$. Note that for $j = 2, \dots, q$, at least one of the vertices v_1, \dots, v_{j-1} is adjacent to v_j . Let us denote such a vertex by $v_{j'}$. Again, $|I_j| \leq 12$, and if we assume $|I_j| = 12$, then at least one of vertices (say, $u_{j''}$) from I_j would have to be a neighbor of $v_{j'}$, which would mean that $u_{j''}$ is already included in one of the sets I_1, \dots, I_{j-1} . Therefore, $|I_j| \leq 11$ for $j = 2, \dots, q$ and we obtain $|I| = \sum_{j=1}^{\gamma_c(G)} |I_j| \leq 12 + 11(\gamma_c(G) - 1) = 11\gamma_c(G) + 1$. Similarly, if $\max\{|N(v) \cap I| : v \in V \setminus I\} \leq \nu$ we have $|I| = \sum_{j=1}^{\gamma_c(G)} |I_j| \leq \nu\gamma_c(G)$. \square

3 A 22-approximate algorithm

Next, we propose an approximation algorithm for the minimum connected dominating set problem in a connected unit-ball graph. The idea is to build a maximal independent set (which is also a dominating set) I , and in the process connect it through intermediate “parent” vertices that will guarantee connectivity. The algorithm consists of three stages: initialization, construction, and improvement, proceeding as follows.

Initialization. This stage initializes the algorithm’s variables. We use the following notations: I denotes the constructed maximal independent set; W denotes the set of vertices adjacent to at least one vertex in I (i.e., $W = N^{\cup}(I)$); U is the set of vertices that are neither in I nor in the neighborhood of I (we will call the vertices in U *unexplored* vertices); and D denotes the connected dominating set sought. We pick the first vertex for the independent set I and the connected dominating set D to be a maximum degree vertex v^* in the graph. Then the remaining variables are initialized correspondingly: $W \leftarrow N(v^*)$ and $U \leftarrow V \setminus (W \cup I)$.

Construction. We continue to construct our connected dominating set D and maximal independent set I by recursively adding an unexplored vertex v^* from the neighborhood of W that has the largest number of unexplored neighbors. In addition, to preserve the connectivity of D we include one of

the explored neighbors u^* of v^* in D and assign u^* to be the *parent* of v^* . When selecting u^* , the preference is given to a neighbor of v^* which is already a parent of another vertex (if such a neighbor exists). This stage terminates when there are no more unexplored vertices, at which point we will obtain a maximal independent set I and a connected dominating set D .

Improvement. Finally, at the improvement stage we attempt to decrease the size of our connected dominating set in case when a new parent vertex was used for each vertex that was added to I at the construction stage, i.e. $|D| = 2|I| - 1$. This is done by adding to D a vertex w^* in W with the largest number of neighbors in I and removing from D parents of all vertices from I that are adjacent to w^* , except for the parent of the first vertex from $N(w^*) \cap I$ that was added to I at the construction stage.

The proposed procedure is summarized in Algorithm 1.

Next, we analyze the performance of Algorithm 1. We will show that our algorithm has a constant performance ratio.

Lemma 3.1 *Algorithm 1 returns a connected dominating set D and a maximal independent set I .*

Proof. We first show that I is a maximal independent set. Indeed, at each step of our algorithm v^* added to I is selected from set U , which does not have any neighbors in I , therefore I is an independent set. Since the algorithm

Algorithm 1. Approximating the minimum connected dominating set problem.

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input a nontrivial, simple, undirected, connected graph  $G = (V, E)$ ;
/* Initialization */
pick  $v^* \in \arg \max\{\deg(v) : v \in V\}$ ;
 $I \leftarrow \{v^*\}$ ;
 $D \leftarrow \{v^*\}$ ;
 $W \leftarrow N(v^*)$ ;
 $U \leftarrow V \setminus (W \cup I)$ ;
/* Construction */
while ( $U \neq \emptyset$ )
    pick  $v^* \in \arg \max\{|N(v) \cap U| : v \in U \cap N^{\cup}(W)\}$ ;
    if ( $D \cap N(v^*) \neq \emptyset$ )
        pick  $u^* \in D \cap N(v^*)$  and set  $parent(v^*) \leftarrow u^*$ ;
    else
        pick  $u^* \in W \cap N(v^*)$  and set  $parent(v^*) \leftarrow u^*$ ;
    end
     $I \leftarrow I \cup \{v^*\}$ ;
     $D \leftarrow D \cup \{v^*\} \cup \{u^*\}$ ;
     $W \leftarrow W \cup (N(v^*) \cap U)$ ;
     $U \leftarrow U \setminus (\{v^*\} \cup N(v^*))$ ;
end
/* Improvement */
if ( $|D| = 2|I| - 1$ )
    pick  $w^* \in \arg \max\{|N(w) \cap I| : w \in W\}$ ;
     $y \leftarrow$  the first vertex in  $N(w^*) \cap I$  added to  $I$ ;
     $D \leftarrow D \cup \{w^*\} \setminus (\{parent(v) : v \in N(w^*) \cap I\} \setminus \{parent(y)\})$ ;
end
return  $D, I$ .

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terminates when $U = \emptyset$, every vertex from V will be either in I or the neighborhood of I , given by W , upon termination. To show that the returned set D is a connected dominating set, first observe that it contains the maximal independent set I , which is also a dominating set. Therefore D is a dominating set. D is connected throughout the execution of the construction stage, since every time we add v^* to D , it is adjacent to its parent vertex u^* , which is also added to D and is such that $G[D \cup \{u^*\}]$ is connected. Finally, we need to show that D is still a connected dominating set after the improvement stage. Let w^* be a vertex from W with the largest number of neighbors in I , and let $N(w^*) \cap I \equiv \{y_1, \dots, y_q\}$ ($q = |N(w^*) \cap I|$) be the list of such neighbors, specified in the order in which vertices y_1, \dots, y_q were added to I at the construction stage. Then y_1 is exactly the vertex y picked at the improvement stage. Note that the improvement stage is executed if and only if each vertex in I has a different parent. Recall that at the construction stage we select a parent for a vertex just added to I so that the preference is given to an existing parent. Hence a different parent for each vertex in $\{y_1, \dots, y_q\}$ is possible if and only if $N(\text{parent}(y_1)) \cap \{y_2, \dots, y_q\} = \emptyset$. This means that $\text{parent}(y_1)$ has to be connected to some vertex from $I \setminus \{y_1, \dots, y_q\}$. Therefore, if we replace $\{\text{parent}(v) : v \in \{y_2, \dots, y_q\}\}$ with w^* in D , the set D will still be connected, since vertices $\{y_1, \dots, y_q\}$ are all connected to w^* and y_1 is connected to the rest of D through $\text{parent}(y_1)$. It also remains a dominating set, since I is still a subset of D . □

Theorem 3.2 *Algorithm 1 has the performance ratio of at most 22.*

Proof. We consider two cases, depending on whether the improvement stage of the algorithm needed to be executed.

- (1) If after termination of the construction stage we have $|D| < 2|I| - 1$, then the improvement stage is not executed and using Lemma 2.2 we obtain

$$|D| \leq 2(|I| - 1) \leq 2(11\gamma_c(G) + 1 - 1) = 22\gamma_c(G).$$

- (2) If after termination of the construction stage we have $|D| = 2|I| - 1$, we run the improvement stage. There are two possibilities:

- (a) $\max\{|N(w) \cap I| : w \in W\} = 2$. Then using Lemma 2.2 with $\nu = 2$ we obtain

$$|D| = 2|I| - 1 \leq 2(2\gamma_c(G)) - 1 < 22\gamma_c(G).$$

- (b) $\max\{|N(w) \cap I| : w \in W\} \geq 3$. Then after performing the improvement stage, we have

$$|D| = 2|I| - 1 - (|N(w^*) \cap I| - 1) + 1 \leq 2(|I| - 1)$$

and using Lemma 2.2 we obtain

$$|D| \leq 2(|I| - 1) \leq 2(11\gamma_c(G) + 1 - 1) = 22\gamma_c(G).$$

Thus, the statement is correct in all possible cases. \square

4 Conclusion

The main result of this paper is a constant-ratio approximation algorithm for the minimum connected dominating set problem in unit-ball graphs. The proposed algorithm was designed specifically with the purpose of proving the approximation result, and perhaps its practical performance can be improved by using more advanced strategies of selecting vertices within the proposed framework, not to mention metaheuristic approaches. Since one of the most interesting applications of the considered problem arises in designing routing protocols for ad hoc wireless networks, developing efficient *distributed* algorithms for approximating minimum connected dominating sets in unit-ball graphs is of great practical interest.

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