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On the Chromatic Number of Graphs

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Abstract. Computing the chromatic number of a graph is an NP-hard problem. For random graphs and some other classes of graphs, estimators of the expected chromatic number have been well studied. In this paper, a new 0–1 integer programming formulation for the graph coloring problem is presented. The proposed new formulation is used to develop a method that generates graphs of known chromatic number by using the KKT optimality conditions of a related continuous nonlinear program.

Key Words. Graph coloring problems, combinatorial optimization, integer programming, test problems.

1. Introduction

Let $G=(V,E)$ be an undirected graph, with vertex set $V(G)=\{1,2,\ldots,n\}$ and edge set $E(G)$; G has no loops or multiple edges. The vertices v_i and v_j are said to be adjacent if there exists an edge $(i,j)\in E(G)$ that is incident to both.

A coloring of G is defined to be an assignment of colors to its vertices so that no pair of adjacent vertices shares identical colors. In other words, a coloring of G is a mapping f of $V(G)$ into $\{1,2,\ldots,n\}$ such that, for any $(i,j)\in E(G)$, $f(i)\neq f(j)$. A coloring induces naturally a partition of $V(G)$ such that the elements of each set in the partition are pairwise nonadjacent; these sets are precisely the subsets of vertices being assigned the same color.

If there exists a coloring of G that uses no more than k colors, G admits a k-coloring (G is k-colorable). The minimal k for which G admits a k-coloring is called the chromatic number and is denoted by $\chi(G)$. Given a

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graph G, the graph coloring problem is the problem of finding the chromatic number $\chi = \chi(G)$ of G as well as a partition of the vertex set induced by a χ-coloring.

Many applications of the graph coloring problem have been identified, such as scheduling and timetabling (Refs. 1–4), which involve constraints expressed as incompatibilities (edges) among certain objects (vertices). Still other problems that have been modeled as graph coloring problems exhibit a wide range of concerns such as computer register allocation (Refs. 5–6), municipal waste collection (Refs. 7–8), mobile radio frequency assignment (Ref. 9), and computation of sparse Jacobian elements (Ref. 10).

The graph coloring problem is an NP-complete problem (Ref. 11); indeed, Garey and Johnson have shown that obtaining colorings using $s\chi(G)$ colors, where $s < 2$, is NP-hard (Ref. 12). Thus, while exact solution procedures exist, the complexity of the problem requires the development of heuristics for large problem instances (Refs. 2–4 and 13–18). For an exhaustive survey of state-of-the-art methods for solving the graph coloring problem, see Ref. 19.

To evaluate the efficiency of a mathematical programming solution in terms of both solution quality and speed, extensive testing of the proposed procedure on a wide variety of problem instances is a crucial phase. Moreover, for the purposes of benchmarking and comparison of different methods, a collection of problems must be available or readily generated. Methods for generating tests problems, as well as collections of test problems in optimization are available; see, for instance, Refs. 20–24. The bibliography due to Floudas and Pardalos (Ref. 20) provides also an extensive list of references in the area.

For the graph coloring problem, there are few types of test problems available. Prior to Leighton in 1979 (Ref. 2), experimental testing of graph coloring heuristics were done mostly on random graphs with at most 100 vertices and unknown chromatic numbers. Comparison with known lower and upper bounds on the chromatic numbers has not proven to be a reliable gauge; also, the application of exact methods to determine the optimal solution is prohibitive even for medium-sized instances. Therefore, the assessment of heuristics was limited to a comparison of colorings that result from the implementation of algorithms, with no knowledge of the gap between even the best heuristic solution and the optimal solution. Leighton (Ref. 2) presented a procedure for generating graphs with known chromatic numbers. Such graphs have been used ever since in computational testing (Refs. 2, 16–18). Prior to discussing the content, motivation, and contribution of this paper, we shall describe briefly some of the types of graphs on which heuristic algorithms have been tested.

1.1. Random Graphs. For a specified edge probability p, $0<p<1$, a random graph $G_{n,p}(V, E)$ with n vertices is a graph such that a pair of vertices has probability p of being adjacent. Such a graph is generated as follows. For every pair of vertices $v_i, v_j \in V$, $i<j$, a uniform random number r_{ij} in the interval [0, 1] is drawn. If $r_{ij} \leq p$, then $(v_i, v_j) \in E(G_{n,p})$. Otherwise, (v_i, v_j) is not in the edge set.

While the chromatic numbers for random graphs are not known, probabilistic estimates for the expected chromatic numbers have been derived (Ref. 25).

1.2. Leighton Graphs. Leighton (Ref. 2) proposed a heuristic method called the recursive largest first (RLF) algorithm and presented a procedure for generating random graphs with known chromatic numbers. An n-vertex Leighton graph G with chromatic number k, where k is a factor of n, is constructed as follows.

(i) Let m be a positive integer such that $m \gg n$, and let k be the greatest common divisor of m and n. Also, let a and c be positive integers such that:

(a) c and m do not have any common divisors;
(b) any prime number that divides m divides $a-1$;
(c) if m is a multiple of 4, then so is $a-1$.

(ii) For a fixed x_0, define $\{x_i\}$, a uniform sequence of random numbers between 0 and $m-1$, by

$$x_i = \text{mod}(ax_{i-1} + c, m), \qquad i = 1, 2, \ldots.$$

(iii) Let $(b_k, b_{k-1}, \ldots, b_2)$ be a random vector of nonnegative integers, with $b_k \geq 1$.

(iv) Let the vertices be denoted by $v_0, v_1, \ldots, v_{n-1}$. Let $v_{y_1}, v_{y_2}, \ldots, v_{y_k}$ form a k-clique. If $b_k > 1$, then add the necessary edges so that $v_{y_{k+1}}, v_{y_{k+2}}, \ldots, v_{y_{2k}}$ likewise form a clique. Continue until b_k k-cliques have been created, and proceed in this manner so that the resulting graph contains b_i i-cliques for $i = k-1, k-2, \ldots, 2$.

Since G contains a k-clique, then $\chi(G) \geq k$. Moreover, the choice of parameters guarantees that the mapping f: $\{v_0, v_1, \ldots, v_{n-1}\} \rightarrow \{1, 2, \ldots, n\}$, given by

$$f(v_{y_i}) = \text{mod}(i, k), \qquad i = 1, 2, \ldots,$$

is a k-coloring of G. The procedure generalizes to the case when k does not divide n.

1.3. Mycielski Graphs. A clique of a graph $G(V, E)$ is a subset C of V such that, for any v_i, $v_j \in C$, we have $(v_i, v_j) \in E$. The clique number $\omega(G)$ is the cardinality of the largest clique in G.

It is immediate that $\chi(G)$ is no less than $\omega(G)$. However, this bound can be quite poor, and graphs have been constructed to illustrate this fact.

The Mycielski graphs (Ref. 15) constitute such a family of graphs. The smallest Mycielski graph M_2 consists of one edge and two vertices. Let M_k with n_k vertices be a Mycielski graph with

$$\chi(M_k) = k, \qquad \omega(M_k) = 2.$$

Then, the succeeding Mycielski graph M_{k+1} is constructed as follows:

(i) $V(M_{k+1}) = V(M_k) \cup \{u_1, u_2, \ldots, u_{n_k}\} \cup \{w\}$;
(ii) $E(M_{k+1}) = E(M_k) \cup \{(u_i, v_j)\colon (v_i, v_j) \in E(M_k)\}_{i=1}^{n_k} \cup \{(w, u_i)\}_{i=1}^{n_k}$.

It can be established that for any $k \geq 2$, the chromatic number of M_k, is k and that M_k contains no 3-clique. Computational testing (Ref. 15) indicates that these provides difficult instances for many heuristic procedures.

1.4. Other Graphs. To test graph coloring heuristics and simulated annealing schemes, Johnson et al. (Ref. 18) implemented algorithms on random graphs with 0.5 edge probability. Suppose that k was the number of colors obtained from the implementation of an algorithm on a specific graph $G_{n,0.5}$. To further test the ability of the procedure, Johnson et al. constructed "cooked" graphs that superficially look like $G_{n,0.5}$, but whose chromatic number is less than k. Suppose that it has been specified that the desired output graph $\hat{G}$ should have chromatic number $\hat{k}$. Then, $\hat{G}$ is created as follows:

(i) Create $\hat{k}$ subsets of vertices by assigning randomly, with equal probability, each of the vertices to a set.
(ii) Any two vertices v_i, v_j which do not belong to the same set will be made adjacent with probability $\hat{k}/2(\hat{k}-1)$.
(iii) Choose a vertex from each set, say $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, and add the edge (v_{i_j}, v_{i_l}) for any $1 \leq j$, $l \leq \hat{k}$, if it has not been created previously.

In the set of techniques which they tested, Johnson et al. found that, for large n, the application of the same algorithm to $\hat{G}$ yielded only a slightly better solution than k, even when $\hat{k}$ is much less than k.

2. New 0–1 Integer Programming Formulation

Given a set of n colors $A=\{1,2,\ldots,n\}$, the graph coloring problem associated with a graph $G=(V,E)$ can be formulated as the following integer program:

$$\text{(P)}\quad \min \sum_{k=1}^{n} y_k,$$

$$\text{s.t.}\quad \sum_{k=1}^{n} x_{ik}=1, \qquad \forall i\in V, \tag{1}$$

$$x_{ik}+x_{jk}\leq 1, \qquad \forall (i,j)\in E, \tag{2}$$

$$y_k\geq x_{ik}, \qquad \forall i\in V, k=1,\ldots,n, \tag{3}$$

$$x_{ik}\in\{0,1\}, \qquad \forall i\in V, k=1,\ldots,n. \tag{4}$$

The optimal objective function value to the above program is the chromatic number of the graph. In the solution, x_{ik} equals 1 if the vertex v_i is colored k, and is zero, otherwise. Moreover, the sets

$$S_k=\{i\in V: x_{ik}=1\},$$

for all k with $y_k>0$, comprise an optimal partition of the vertices. This paper suggests a method of generating graphs of known chromatic numbers which uses the optimality conditions of a nonlinear program. A user inputs the size of the graph desired by specifying the cardinality of the vertex set and the density of the graph. A graph $G=(V,E)$ is generated randomly with the desired specifications. We will use approximate colorings $\tilde{C}\geq\chi(G)$ of the derived graphs G_i of G until a graph G_i is determined such that $\tilde{C}=\chi(G_i)$. In more detail, G_i is obtained from G_{i-1} by removing randomly a certain number of edges. At each step, a continuous nonlinear program will be used to determine if $\tilde{C}_i=\chi(G_i)$. After obtaining $G_{i'}=(V,E')$, edges will be added in such a way that $\chi(G_{i'})$ is unaffected. The final result will be a graph $\bar{G}=(V,\bar{E})$ such that $\chi(\bar{G})=\chi(G_{i'})$ and the density of $\bar{G}$ is approximately equal to the density of G.

3. Main Step

Assume that a heuristic procedure has been used in the graph under consideration to obtain $\tilde{C}$. An approximate coloring $\tilde{C}$ determines a feasible solution $(\tilde{x},\tilde{y})$ to P, namely,

$$\tilde{x}_{ik}=\begin{cases}1, & \text{if } v_i \text{ is colored } k,\\ 0, & \text{otherwise,}\end{cases}$$

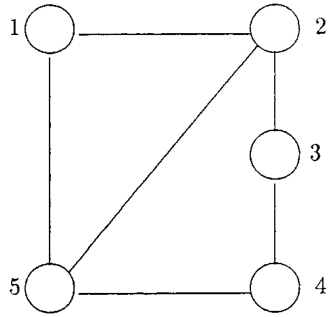

Fig. 1. Small example: A graph of 5 vertices and 6 arcs.

and

$$y_1 = y_2 = \cdots = y_{\tilde{C}} = 1,$$

with

$$S_k = \{i: \tilde{x}_{ik} = 1\}, \qquad k = 1, 2, \ldots, \tilde{C}.$$

$(\tilde{x}, \tilde{y})$ is used to define a nonlinear program $\bar{\text{P}}$ and a candidate optimal solution to $\bar{\text{P}}$, which will have less variables than P, since the coloring index will go up to $\tilde{C}$.

Figure 1 is an example graph to which we will refer throughout. Suppose that a heuristic was applied to the graph G drawn in Fig. 1 and that the result obtained was $\tilde{C} = 4$ with

$$S_1 = \{1, 4\}, \qquad S_2 = \{2\}, \qquad S_3 = \{3\}, \qquad S_4 = \{5\}.$$

Table 1 gives the variables associated with $\bar{\text{P}}$.

In $\bar{\text{P}}$, we intend to replace the binary constraints (4) by nonnegativity constraints and add to the objective a penalty function of $x_{ij}(x_{ij} - 1)$ for each variable x_{ij}. In more detail, let M be a large positive number. Then, $\bar{\text{P}}$

Table 1. Variables associated with Problem $\bar{\text{P}}$.

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}
x_{41}	x_{42}	x_{43}	x_{44}
x_{51}	x_{52}	x_{53}	x_{54}
y_1	y_2	y_3	y_4

is formulated as follows:

$$(\bar{\text{P}}) \quad \min \quad F(x,y) = \sum_{k=1}^{\tilde{C}} \left\{ y_k + M \sum_{i=1}^{n} [x_{ik}(x_{ik}-1)]^2 \right\}$$

$$\text{s.t.} \quad \sum_{k=1}^{\tilde{C}} x_{ik} = 1, \qquad \forall i \in V, \tag{5}$$

$$x_{ik} + x_{jk} \leq 1, \qquad \forall (i,j) \in E, \tag{6}$$

$$y_k \geq x_{ik}, \qquad \forall i \in V, k = 1, \ldots, \tilde{C}, \tag{7}$$

$$x_{ik} \geq 0, \qquad \forall i \in V, k = 1, \ldots, \tilde{C}. \tag{8}$$

Suppose that vertex i is colored by using the color $[i] \in \{1, 2, \ldots, \tilde{C}\}$. Then, the objective function F can be rewritten as

$$\min F(x,y) = \sum_{i=1}^{n} \left\{ y_{[i]}/|S_{[i]}| + M \sum_{k=1}^{\tilde{C}} [x_{ik}(x_{ik}-1)]^2 \right\},$$

where $|S_{[i]}|$ refers to the cardinality of the independent set to which the vertex i belongs. $(\tilde{x}, \tilde{y})$ will be used to define a candidate optimal solution (x', y') to $\bar{\text{P}}$. Theorem 4.3.7 of Bazaraa and Shetty (Ref. 26) will be used to determine if (x', y') is optimal to $\bar{\text{P}}$. As it will be shown, if (x', y') is optimal to $\bar{\text{P}}$, then $\tilde{C} = \chi(G)$.

4. Properties of Problem $\bar{\text{P}}$

The major difference between Problems P and $\bar{\text{P}}$ is that $x_{ij} \in \{0, 1\}$ in P, whereas $x_{ij} \geq 0$ in $\bar{\text{P}}$. Note that the constraints (5) together with nonnegativity imply that $0 \leq x_{ij} \leq 1$ for $\bar{\text{P}}$. We attempt to make x_{ij} almost integer by controlling the objective function. In fact, denoting the optimal objective function to $\bar{\text{P}}$ by F^*, the following is immediate:

$$F^* \leq \chi(G).$$

This is true, since $x_{ij} \in \{0, 1\}$ if and only if $x_{ij}(x_{ij} - 1) = 0$ and any solution to P is feasible to $\bar{\text{P}}$; thus, $F^* \leq \chi(G)$.

Consider the univariate function

$$h(u) = [u(u-1)]^2, \qquad 0 \leq u \leq 1,$$

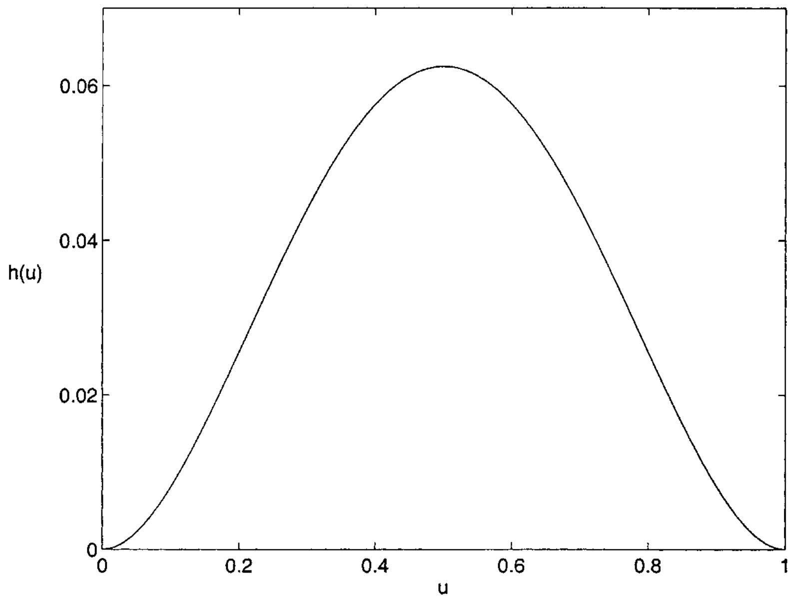

Fig. 2. Univariate function $h(u)=[u(u-1)]^2$, $0\leq u\leq 1$.

shown in Fig. 2. It has a single maximum value corresponding to $u=0.5$ and two minima at 0 and 1. Furthermore, $h(u)$ is strictly convex for u belonging to the intervals $[0, 0.2113)$ and $(0.7887, 1]$ and is concave otherwise. $h(u)$ is symmetric about the line $u=0.5$; i.e.,

$$h(u)=h(1-u).$$

It is also easy to establish that

$$h'(u)=-h'(1-u),$$

where $h'(u)$ denotes the first derivative of $H(u)$.

Lemma 4.1. Let T be any constant such that $T>\tilde{C}\geq\chi(G)\geq 2$ and $T>5$. Let (x, y) be feasible to $\bar{\mathrm{P}}$ such that $1/T<x_{ij}<(T-1)/T$ for at least one pair i and j. Then, an $M\geq T^5$ ensures that (x, y) is not optimal to $\bar{\mathrm{P}}$.

Proof. Observe that

$$\begin{aligned} F(x, y) &> M[x_{ij}(x_{ij}-1)]^2 \\ &> T>\tilde{C} \\ &\geq\chi(G)\geq F^*. \end{aligned}$$

Note that, for $T > 5$, the positive components x_{ij} of x in an optimal solution will lie in the intervals where $[x_{ij}(x_{ij} - 1)]^2$ is convex. □

Lemma 4.2. Let M be chosen as above. If (x, y) is a global solution to $\bar{\text{P}}$, then exactly one $x_{ij} \in \{x_{ik}\}_{k=1}^{\tilde{C}}$ is in $[(1 - 1/T), 1]$ for every i.

Proof. The pair (x, y) must satisfy the first set of constraints (5) of the formulation of $\bar{\text{P}}$; i.e.,

$$\sum_{k=1}^{\tilde{C}} x_{ik} = 1, \qquad \forall i \in V.$$

By the choice of M, the previous lemma implies that all $\{x_{ik}\}_{k=1}^{\tilde{C}}$ are in $[0, 1/T]$ or $[(1 - 1/T), 1]$. Suppose that, for a given i, all $\{x_{ik}\}_{k=1}^{\tilde{C}}$ are in $[0, 1/T]$. Then,

$$T > \tilde{C} \Rightarrow 1/T < 1/\tilde{C} \Rightarrow \sum_{k=1}^{\tilde{C}} x_{ik} < 1.$$

On the other hand, suppose that more than one element in the set $\{x_{ik}\}_{k=1}^{\tilde{C}}$ is in $[(1 - 1/T), 1]$; then,

$$\sum_{k=1}^{\tilde{C}} x_{ik} > 1,$$

since $T \geq 3$. Thus, for each i, exactly one element of $\{x_{ik}\}_{k=1}^{\tilde{C}}$ is in the interval $[(1 - 1/T), 1]$. □

Let I_y denote the index set of y_k, $k = 1, 2, \ldots, \tilde{C}$, such that $y_k \in [(1 - 1/T), 1]$.

Lemma 4.3. Let M be chosen as above. Then, for an optimal solution (x, y) to $\bar{\text{P}}$, $|I_y|$ is equal to $\chi(G)$.

Proof. Since we are dealing with a minimization problem, it can be assumed that

$$y_k = \max_i \{x_{ik}\}.$$

Suppose that $|I_y| < \chi(G)$. Without loss of generality, assume that $I_y = \{1, 2, \ldots, a\}$. Therefore, there are exactly a sets $\{x_{ik}\}_1^n, k = 1, 2, \ldots, a$ (see columns in Table 1), which contain elements in $[(1 - 1/T), 1]$. Note that

$$x_{ik}^* = 1, \qquad \text{if } x_{ik} > (1 - 1/T),$$

$$x_{ik}^* = 0, \qquad \text{otherwise.}$$

Since (x, y) satisfies the constraints (5)–(7) of the mathematical formulation of $\bar{\mathrm{P}}$, then (x^*, y^*) is feasible for P.

Since

$$y_k = 1, \qquad k = 1, 2, \dots, a,$$
$$y_k = 0, \qquad k = a+1, a+2, \dots, \tilde{C},$$

we have

$$\sum y_k^* < \chi(G).$$

On the other hand, suppose that $|I_y| \geq \chi(G) + 1$. Then,

$$\sum_{i=1}^{\tilde{C}} y_i > [\chi(G)+1](1 - 1/T) = \chi(G) + 1 - [\chi(G)+1]/T,$$

and

$$\begin{aligned} &T > \tilde{C} \geq \chi(G) \\ \Rightarrow\ &T \geq \chi(G) + 1 \\ \Rightarrow\ &1 \geq [\chi(G)+1]/T \\ \Rightarrow\ &0 \leq 1 - [\chi(G)+1]/T. \end{aligned}$$

Thus,

$$\sum_{i=1}^{\tilde{C}} y_i > \chi(G) \geq F^*,$$

contradicting the fact that (x, y) was assumed optimal. □

Definition 4.1. Let $\epsilon > 0$. An ϵ-optimal solution (x', y') is a solution feasible for P and is such that

$$F(x', y') - F^* < \epsilon.$$

Lemma 4.4. Let M be chosen as above and let $\epsilon < 1 - (n+1)/T$. Then, $|I_{y'}|$ in an ϵ-solution (x', y') to $\bar{\mathrm{P}}$ is less than or equal to $\chi(G)$.

Proof. In an optimal solution (x, y), we have:

$$F^* = F(x, y) \leq \chi(G).$$

Suppose that (x', y') is an ϵ-optimal solution such that

$$|I_{y'}| \geq \chi(G) + 1.$$

Then,

$$F(x', y') \geq [\chi(G) + 1](1 - 1/T).$$

Thus,

$$\begin{aligned} F(x', y') - F(x, y) &\geq [\chi(G) + 1](1 - 1/T) - F(x, y) \\ &\geq [\chi(G) + 1](1 - 1/T) - \chi(G) \\ &= 1 - (n + 1)/T, \end{aligned}$$

indicating that (x', y') is not an ϵ-solution.

Suppose that a partition of the vertices has been obtained and a solution $(\tilde{x}, \tilde{y})$ has been determined from the partition. Define (x', y') as follows.

Each vertex i is associated with a set of variables $\{\tilde{x}_{ik}\}_{k=1}^{\tilde{C}}$. Each vertex is assigned precisely one color by the set of constraints (1). As before, let $[i] \in \{1, \ldots, \tilde{C}\}$ denote the color of vertex i. Let m_i be the minima of the following functions:

$$x_{i[i]}/|S_{[i]}| + M[x_{i[i]}(x_{i[i]} - 1)]^2,$$

in the interval $[(1 - 1/T), 1]$. Then, for each i,

$$x'_{ik} = \begin{cases} m_i, & \text{if } k = [i], \\ (1 - m_i)/(\tilde{C} - 1), & k = 1, 2, \ldots, \tilde{C}, k \neq [i]. \end{cases}$$

Consider the following nonlinear program:

$$\begin{aligned} \min \quad & f(x), \\ \text{s.t.} \quad & g_i(x) \leq 0, \qquad \text{for } i = 1, 2, \ldots, m, \\ & h_i(x) \leq 0, \qquad \text{for } i = 1, 2, \ldots, l, \\ & x \in X, \end{aligned}$$

where $f: \mathscr{R}^n \to \mathscr{R}$, $g_i: \mathscr{R}^n \to \mathscr{R}$, for $i = 1, 2, \ldots, m$, and $h_i: \mathscr{R}^n \to \mathscr{R}$, for $i = 1, 2, \ldots, l$. It is well known (Ref. 26) that the above program can be associated with the following Kuhn–Tucker system of equations at $\bar{x}$:

$$\nabla f(\bar{x}) + \nabla g(\bar{x})u + \nabla h(\bar{x})v = 0,$$

$$u^t g(\bar{x}) = 0,$$

$$u \geq 0,$$

where $\nabla g(\bar{x})$ is an $n \times m$ matrix and $\nabla h(\bar{x})$ is an $n \times l$ matrix whose ith columns are $\nabla g_i(\bar{x})$ and $\nabla h_i(\bar{x})$, respectively. The vectors u and v are known as the Lagrangian multipliers.

Construct the Kuhn–Tucker system associated with $\bar{\mathrm{P}}$ at (x', y'). It is easy to show that, if there exist Lagrangian multipliers that solve the Kuhn–Tucker system, then (x', y') is an ϵ-optimal solution to $\bar{\mathrm{P}}$ with $\epsilon < 1/T$. Since

$$1/T < 1 - (n+1)/T, \qquad \text{for } T > n+2,$$

Lemma 4.4 implies that

$$|I_{y'}| \leq \chi(G).$$

By construction of (x', y'), it must be that $(\tilde{x}, \tilde{y})$ gives an optimal coloring of the graph under construction. □

5. Illustrative Example

Consider the graph

$$G_1 = (V_1, E_1), \qquad V_1 = \{1, 2, 3, 4, 5\}$$

in Fig. 3, and suppose that a heuristic has been applied on G obtaining an approximate chromatic number $\tilde{C} = 4$ and the corresponding graph partition is the following:

$$S_1 = \{1, 4\}, \qquad S_2 = \{2\}, \qquad S_3 = \{3\}, \qquad S_4 = \{5\}.$$

The candidate solution (x', y') to the derived $\bar{\mathrm{P}}$ is computed as follows. Since Vertex 1 is colored by using Color 1,

$$x'_{11} = m_1 = \min x/|S_{[1]}| + M[x(x-1)]^2,$$

where $|S_{[1]}| = 2$ and M is chosen equal to 16, since for this example, $M > T^5 \geq \tilde{C}$ is a large overestimation. Therefore,

$$m_1 = 0.9836 \Rightarrow x'_{11} = 0.9836,$$

$$x'_{1k} = (1 - m_1)/(\tilde{C} - 1) = 0.0055, \qquad k = 2, 3, 4.$$

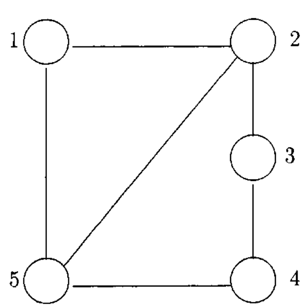

Fig. 3. Input graph G_1.

Since Vertex 2 is colored by using Color 2,

$$x'_{22} = m_2 = \min x/|S_{[2]}| + M[x(x-1)]^2,$$

where $|S_{[2]}| = 1$ and $M = 16$. Therefore,

$$m_2 = 0.9652 \Rightarrow x'_{22} = 0.9652,$$

$$x'_{2k} = (1 - m_2)/(\tilde{C} - 1) = 0.0116, \qquad k = 1, 3, 4.$$

Since Vertex 3 is colored by using Color 3, we have

$$x'_{33} = m_3 = \min x/|S_{[3]}| + M[x(x-1)]^2,$$

where $|S_{[3]}| = 1$ and $M = 16$. Therefore,

$$m_3 = 0.9652 \Rightarrow x'_{33} = 0.9652,$$

$$x'_{3k} = (1 - m_3)/(\tilde{C} - 1) = 0.0116, \qquad k = 1, 2, 4.$$

Since Vertex 4 is colored by using Color 1,

$$x'_{41} = m_4 = m_1 = 0.9836,$$

$$x'_{4k} = (1 - m_1)/(\tilde{C} - 1) = 0.0055, \qquad k = 2, 3, 4.$$

Since Vertex 5 is colored by using Color 4,

$$x'_{54} = m_5 = \min x/|S_{[4]}| + M[x(x-1)]^2,$$

where $|S_{[4]}| = 1$ and $M = 16$. Therefore,

$$m_5 = 0.9652 \Rightarrow x'_{54} = 0.9652,$$

$$x'_{5k} = (1 - m_5)/(\tilde{C} - 1) = 0.0116, \qquad k = 1, 2, 3.$$

Finally,

$$y_k = \max_{l=1}^{n}\{x_{lk}\}.$$

Therefore,

$$y'_1 = 0.9836 \quad \text{and} \quad y'_2 = y'_3 = y'_4 = 0.9652.$$

Unfortunately, a solution to the Kuhn–Tucker system associated to $\bar{\mathrm{P}}$ at (x', y') so obtained cannot be found.

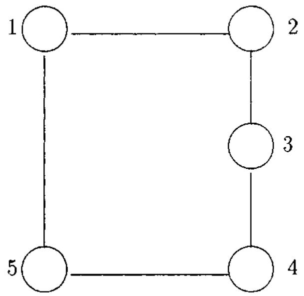

Fig. 4. Modified graph G_2 after dropping edge (2, 5) from graph G_1.

From the graph G_1 the edge (2, 5) is dropped. The resulting new graph G_2 is drawn in Fig. 4. Suppose that a heuristic is newly applied on G_2 and that the result obtained is $\tilde{C} = 3$ with the following corresponding graph partition:

$$S_1 = \{1, 4\}, \qquad S_2 = \{2, 5\}, \qquad S_3 = \{3\}.$$

As above, the new candidate solution (x', y') to the derived $\bar{\text{P}}$ is computed as follows.

Since Vertex 1 is colored by using Color 1,

$$x'_{11} = m_1 = \min x/|S_{[1]}| + M[x(x-1)]^2,$$

where $|S_{[1]}| = 2$ and $M = 16$. Therefore,

$$m_1 = 0.9836 \Rightarrow x'_{11} = 0.9836,$$

$$x'_{1k} = (1 - m_1)/(\tilde{C} - 1) = 0.0055, \qquad k = 2, 3.$$

Since Vertex 2 is colored by using Color 2,

$$x'_{22} = m_2 = \min x/|S_{[2]}| + M[x(x-1)]^2,$$

where $|S_{[2]}| = 2$ and $M = 16$. Therefore,

$$m_2 = 0.9652 \Rightarrow x'_{22} = 0.9836,$$

$$x'_{2k} = (1 - m_2)/(\tilde{C} - 1) = 0.0055, \qquad k = 1, 3.$$

Since Vertex 3 is colored by using Color 3,

$$x'_{33} = m_3 = \min x/|S_{[3]}| + M(x(x-1))^2,$$

where $|S_{[3]}| = 1$ and $M = 16$. Therefore,

$$m_3 = 0.9652 \Rightarrow x'_{33} = 0.9652,$$

$$x'_{3k} = (1 - m_3)/(\tilde{C} - 1) = 0.0116, \qquad k = 1, 2.$$

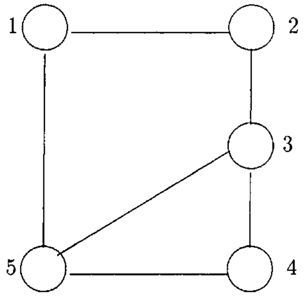

Fig. 5. Final graph G_3 having chromatic number 3.

Since Vertex 4 is colored by using Color 1,

$$x'_{41} = m_4 = m_1 = 0.9836,$$

$$x'_{4k} = (1 - m_1)/(\tilde{C} - 1) = 0.0055, \qquad k = 2, 3.$$

Since Vertex 5 is colored by using Color 2,

$$x'_{52} = m_5 = m_2 = 0.9836,$$

$$x'_{5k} = (1 - m_2)/(\tilde{C} - 1) = 0.0116, \qquad k = 1, 3.$$

Finally,

$$y_k = \max_{l=1}^{n} \{x_{lj}\}.$$

Therefore,

$$y'_3 = 0.9652 \quad \text{and} \quad y'_1 = y' = 0.9836.$$

In this case, a solution to the Kuhn–Tucker system associated with $\bar{\mathrm{P}}$ at (x', y') is found implying $\chi(G_2) = 3$. An edge is added to G_2 without affecting $\chi(G_2)$. The final outcome is the graph G_3 drawn in Fig. 5. It has five nodes, six edges, and chromatic number 3.

6. Conclusions

For many discrete optimization problems, a continuous formulation can be found, from which new properties and efficient algorithms can result. While a typical combinatorial method generates a sequence of states representing a partial solution, a continuous approach for solving discrete optimization problems is based on different equivalent characterizations in a larger and continuous space. These characterizations include continuous relaxations or continuous formulations.

In this paper, a new 0–1 integer programming formulation for the graph coloring problem is presented and a KKT-based method is described to construct test problems.

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